A variant of a theorem of C. Voisin

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0 Introduction

Let S be a smooth projective surface defined over $\mathbb{C}$, and let $A_0(S)$ be the group of zero cycles of degree zero on S modulo rational equivalence. The n–fold symmetric product $S^{(n)}$ parametrises effective zero–cycles of degree n on S. One says that $A_0(S)$ is finite dimensional if there exists $n_0 \in \mathbb{N}$ such that the map

$$f_{n,n} : S^{(n)} \times S^{(n)} \to A_0(S)$$

defined by $f_{n,n}(A, B) = [A - B]$ is surjective for all $n \geq n_0$. Mumford [Mu] proved that $A_0(S)$ is not finite dimensional if $p_g(S) > 0$. Let $C \subset S$ be a smooth curve, and let $i : C \to S$ be the inclusion morphism. If the induced map $i_* : A_0(C) \to A_0(S)$ is surjective, then $A_0(S)$ is finite dimensional and $A_0(S) \cong \mathrm{Alb}(S)$; see [R1, Thm. 4]. Hence if the map i_* is surjective, its kernel coincides with the kernel of the map $J(C) \to \mathrm{Alb}(S)$, which is the the subtorus

$$J(C)_{\mathrm{var}} = H^1_{\mathrm{var}}(C, \mathbb{C})/F^1 H^1_{\mathrm{var}}(C, \mathbb{C}) + H^1_{\mathrm{var}}(C, \mathbb{Z})$$

associated to $H^1_{\mathrm{var}}(C) = \ker(i_* : H^1(C) \to H^3(S))$.

Voisin has studied the kernel of i_* for smooth surfaces $S \subset \mathbb{P}^3$ of degree d. In this case one has $\mathrm{Alb}(S) = 0$, hence $\mathrm{Tors}(A_0(S)) = 0$ by Roitman's theorem [R2]. It follows that $\mathrm{Tors}(A_0(C)) \subseteq \ker i_*$. Voisin showed that equality holds if S is a very general surface of degree $d \geq 5$ and $C \subset S$ is a smooth plane section; see [V1, Thm. 2.1].

Let us consider curves that are obtained by intersecting the surface $S \subset \mathbb{P}^3$ with a smooth surface $Y \subset \mathbb{P}^3$ of degree $e > 1$. Set

$$\mathrm{CH}^1(Y)_0 = \{z \in \mathrm{CH}^1(Y)_{\mathbb{Q}} | \deg(z|_C) = 0\}$$
$$\mathrm{CH}^2(\mathbb{P}^3)_0 = \{z \in \mathrm{CH}^2(\mathbb{P}^3)_{\mathbb{Q}} | \deg(z|_S) = 0\}$$

and note that the second group is zero. Let $j : C \to S$ be the inclusion morphism. The commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^1(Y)_0 & \to & \mathrm{CH}^2(\mathbb{P}^3)_0 = 0 \\ \Big\downarrow{\scriptstyle j^*} & & \Big\downarrow \\ A_0(C)_{\mathbb{Q}} & \xrightarrow{i_*} & A_0(S)_{\mathbb{Q}} \end{array}$$

shows that $j^* \mathrm{CH}^1(Y)_0 \subseteq \ker(i_* : A_0(C)_{\mathbb{Q}} \to A_0(S)_{\mathbb{Q}})$. As the map $j^* : \mathrm{CH}^1(Y)_0 \to A_0(C)_{\mathbb{Q}}$ is injective if C is very general and if the degree d is sufficiently large (Proposition 1.9), we may have $\ker i_* \supsetneqq \mathrm{Tors}(A_0(C))$ in this case (For instance, let $Y \subset \mathbb{P}^3$ be a smooth quadric and let $\xi = L_1 - L_2 \in \mathrm{CH}^1(Y)_0$ the difference of two lines from the different rulings.)

The purpose of this note is to extend Voisin's theorem to the case of hypersurface sections; we show that the kernel of $i_* : A_0(C)_{\mathbb{Q}} \to A_0(S)_{\mathbb{Q}}$ equals $j^* \mathrm{CH}^1(Y)_0$ if S is very general and $d = \deg S$ is sufficiently large with respect to $e = \deg Y$. (It is possible to obtain precise degree bounds; see Remark 2.2 (ii).) More generally we consider surfaces S defined by global sections of an ample line bundle L on a smooth projective threefold W and curves $C \subset S$ obtained by intersecting S with an ample divisor $Y \subset W$. We show that there is a certain subgroup $\mathrm{CH}^1(Y)_{\mathrm{var}} \subseteq \mathrm{CH}^1(Y)_0$ such that $\ker i_* \subseteq j^* \mathrm{CH}^1(Y)_{\mathrm{var}}$ if L is sufficiently ample; see Theorem 2.1. The inclusion $\ker i_* \subseteq j^* \mathrm{CH}^1(Y)_{\mathrm{var}}$ is an equality if W is a Fano threefold, but probably not in general; see Remark 2.2 (i).

In her paper Voisin uses an infinitesimal invariant δZ associated to a relative zero cycle $Z_T \in \mathrm{CH}_0(S_T/T)$. In our case it is more convenient to use a connectivity theorem for two families U_T, V_T of quasi–projective varieties over T, following the ideas of Nori [N]. Both methods are essentially equivalent (cf. the footnotes in [V1] on pages 79 and 85). We prove the connectivity theorem, and some other technical results, in Section 1. In Section 2 we prove the proposed variant of Voisin's theorem.

Voisin's theorem has been extended in a different direction (the case of higher–dimensional varieties) by M. Asakura and S. Saito [AS]. The problem considered in this note was suggested to me by Prof. J.P. Murre. I would like to thank Professors Murre and Peters for comments and useful discussions.

Notation and conventions. For an abelian group G we write $G_{\mathbb{Q}} = G \otimes \mathbb{Q}$. Cohomology is taken with $\mathbb{C}$–coefficients, unless stated otherwise. We say that a property (P) holds for a *very general* element of a topological space X if the set of elements of X that do not satisfy property (P) is a countable union of proper closed subsets. We say that a property (P) holds for *sufficiently ample* line bundles if there exists a line bundle L_0 such that property (P) holds for all line bundles L such that $L \otimes L_0^{-1}$ is ample.

1 Connectivity theorem

Let W be a smooth projective threefold, $Y \subset W$ a smooth ample divisor and L an ample line bundle on W such that $H^0(W, L) \neq 0$. For $t \in H^0(W, L)$ we write $S_t = V(t)$, $C_t = Y \cap S_t$. Set

$$\begin{aligned} G &= \{g \in \mathrm{Aut}(W) | g.Y \subset Y\} \\ \Delta &= \{[t] \in \mathbb{P}H^0(W, L) | C_t \text{ is singular}\} \\ B &= (\mathbb{P}H^0(W, L) \setminus \Delta)/G. \end{aligned}$$

Set $W_B = W \times B$, $Y_B = Y \times B$. Let $S_B \to B$ be the universal family of surfaces in W defined by sections of L and set $C_B = S_B \cap Y_B$. Let $h : T \to B$ be a finite étale morphism. Set $U_T = W_T \setminus Y_T$, $V_T = S_T \setminus C_T$.

Lemma 1.1. *If L is sufficiently ample then*

(i) $H^k(W_T, S_T) = 0$ *for* $k \leq 4$*;*

(ii) $H^k(Y_T, C_T) = 0$ *for* $k \leq 2$.

Proof: This follows from results of Nori [N], more specfically a version of [N, Thm. 4] that can obtained from Prop. 3.1 and Lemmas 2.1 and 2.2 of [loc. cit.]. □

Corollary 1.2. *If L is sufficiently ample then $H^k(U_T, V_T) = 0$ for $k \leq 3$.*

Proof: Use Lemma 1.1 and the exact sequence

$$H^k(W_T, S_T) \to H^k(U_T, V_T) \to H^{k-1}(Y_T, C_T).$$

□

In Section 2 we shall also need the vanishing of $H^4(U_T, V_T)$. By Lemma 1.1, this group injects into $H^3(Y_T, C_T)$. As the latter group is nonzero even if L is sufficiently ample (cf. [N, p. 352]), the vanishing of $H^4(U_T, V_T)$ does not follow from Nori's results; we shall prove it in this section, using the techniques of [N] and infinitesimal computations. The divisor $D_T = S_T \cup Y_T$ is a divisor with relative normal crossings. Following [DI, (4.2.1.2)] we define

$$\Omega^\bullet_{W_T}(S_T, Y_T) = \Omega^\bullet_{W_T}(\log D_T)(-S_T)$$
$$\Omega^\bullet_{W_T}(Y_T, S_T) = \Omega^\bullet_{W_T}(\log D_T)(-Y_T).$$

Using the exact sequence

$$0 \to \Omega^\bullet_{W_T}(S_T, Y_T) \to \Omega^\bullet_{W_T}(\log Y_T) \to \Omega^\bullet_{S_T}(\log C_T) \to 0$$

and the five lemma, one shows that

$$H^k(U_T, V_T) \cong \mathbb{H}^k(\Omega^\bullet_{W_T}(S_T, Y_T)).$$

Lemma 1.3. *If* $\mathbb{H}^4(\sigma_{\geq 2}\Omega^\bullet_{W_T}(S_T, Y_T)) = 0$ *then* $H^4(U_T, V_T) = 0$.

Proof: Define a filtration $G^\bullet$ on $H^4(U_T, V_T)$ by

$$G^p H^4(U_T, V_T) = \mathrm{im}(\mathbb{H}^4(\sigma_{\geq p}\Omega^\bullet_{W_T}(S_T, Y_T) \to \mathbb{H}^4(\Omega^\bullet_{W_T}(S_T, Y_T)).$$

As in [N, §1] one shows that $G^\bullet$ is coarser than the Hodge filtration $F^\bullet$ on $H^4(U_T, V_T)$. We obtain $F^2 H^4(U_T, V_T) = 0$, hence $H^4(U_T, V_T) = 0$ because $\mathrm{Gr}_i^W H^k(U_T, V_T) = 0$ if $i < 4$. □

Let $f : W_T \to T$ be the projection map. Set

$$\mathcal{H}^k_{U,V} = \mathbb{R}^k f_* \Omega^\bullet_{W_T/T}(S_T, Y_T).$$

The sheaf $\mathcal{H}^k_{U,V}$ is filtered by subsheaves $\mathcal{F}^m\mathcal{H}^k_{U,V}$, with graded pieces $\mathcal{H}^{p,q}_{U,V}$. Using [DI, Cor. 4.2.4] and [D, Thm. (5.5)] one shows that the sheaves $\mathcal{H}^k_{U,V}$ and $\mathcal{H}^{p,q}_{U,V}$ are locally free and that $\mathcal{H}^{p,q}_{U,V} \otimes k(t) \cong H^q(W, \Omega^p_W(S_t, Y))$ for all $t \in T$. The de Rham complex $\Omega^\bullet(\mathcal{H}^k_{U,V})$ is filtered by subcomplexes

$$\Omega^\bullet(\mathcal{F}^m\mathcal{H}^k_{U,V}) : \mathcal{F}^m\mathcal{H}^k_{U,V} \xrightarrow{\nabla} \Omega^1_T \otimes \mathcal{F}^{m-1}\mathcal{H}^k_{U,V} \to \cdots$$

for every $m \in \mathbb{N}$.

Lemma 1.4. *We have*

(i) $H^k(U, V_t) = 0$ *if* $k \leq 2$;

(ii) *if* $Y \subset W$ *is an ample divisor then* $H^k(U, V_t) = 0$ *if* $k \neq 3$.

Proof: (i): As $S_t \subset W$ is an ample divisor, $W \setminus S_t$ is an affine variety of dimension 3. Hence it has the homotopy type of a CW complex of dimension ≤ 3 (cf. [Mi, §7]) and $H^k(W, S_t) \cong H_{6-k}(W \setminus S_t) = 0$ if $k < 3$. In a similar way one shows that $H^k(Y, C_t) = 0$ if $k < 2$, and the statement follows from the exact sequence $H^k(W, S_t) \to H^k(U, V_t) \to H^{k-1}(Y, C_t)$.

(ii): If $Y \subset W$ is ample, $U = Y \setminus W$ and $V_t = S_t \setminus C_t$ are affine varieties, hence [loc. cit.] $H^k(U) = 0$ if $k > 3$ and $H^k(V_t) = 0$ if $k > 2$. The statement then follows from the exact sequence of relative cohomology. □

Let $\Sigma_{W,L}$ be the bundle of differential operators of order ≤ 1 on sections of L. If we pull back the extension

$$0 \to \mathcal{O}_W \to \Sigma_{W,L} \to T_W \to 0$$

along the map $T_W(-\log Y) \to T_W$, we obtain an extension

$$0 \to \mathcal{O}_W \to \Sigma_{W,L}(-\log Y) \to T_W(-\log Y) \to 0.$$

Contraction with the 1–jet $j^1(t)$ of $t \in H^0(W, L)$ defines an exact sequence

$$0 \to T_W(-\log D_t) \to \Sigma_{W,L}(-\log Y) \to L \to 0. \tag{1}$$

We construct a Jacobi ring R' using the method of Green [G1]. Define $J'_{W,t}(K_W^a \otimes L^b)$, for $(a, b) \neq (1, 1)$, as the image of the map

$$H^0(W, K_W^a \otimes \Sigma_{W,L}(-\log Y) \otimes L^{b-1}) \to H^0(W, K_W^a \otimes L^b)$$

induced by the exact sequence (1). We put $J'_{W,t}(K_W \otimes L) = 0$ and

$$R'_{W,t}(K_W^a \otimes L^b) = H^0(W, K_W^a \otimes L^b)/J'_{W,t}(K_W^a \otimes L^b).$$

As in [G2, p. 43] one shows that

$$H^p(W, \Omega_W^{3-p}(Y, S_t)) \cong R'_{W,t}(K_W \otimes L^{p+1})$$

if L is sufficiently ample. The exact sequence (1) induces a logarithmic Kodaira–Spencer map

$$\rho : A = H^0(W, L) \to H^1(W, T_W(-\log D_t)).$$

Lemma 1.5. *Set $A = H^0(W, L)$. If L is sufficiently ample, the complex*

$$0 \to \mathcal{H}^{b,3-b}_{U,V} \to A^\vee \otimes \mathcal{H}^{b-1,4-b}_{U,V} \to \bigwedge^2 A^\vee \otimes \mathcal{H}^{b-2,5-b}_{U,V} \tag{2}$$

is exact for $b \geq 2$.

Proof: We shall verify the exactness of

$$0 \to \mathcal{H}^{2,1}_{U,V} \xrightarrow{\overline{\nabla}} A^\vee \otimes \mathcal{H}^{1,2}_{U,V} \xrightarrow{\overline{\nabla}} \bigwedge^2 A^\vee \otimes \mathcal{H}^{0,3}_{U,V},$$

the other cases being similar. It suffices to show that for every $t \in A$ the complex of fibers over t is exact. Using the duality induced by the perfect pairing

$$H^p(W, \Omega^{3-p}_W(S_t, Y)) \otimes H^{3-p}(W, \Omega^p_W(Y, S_t)) \to H^3(W, K_W) \cong \mathbb{C}$$

we reduce to proving the exactness of the complex

$$\bigwedge^2 A \otimes H^0(\Omega^3_W(Y, S_t)) \to A \otimes H^1(\Omega^2_W(Y, S_t)) \to H^2(\Omega^1_W(Y, S_t)) \to 0. \tag{3}$$

The maps in the complex (3) are given by cup product with the logarithmic Kodaira–Spencer class, followed by contraction. As the logarithmic Kodaira–Spencer class corresponds to the extension class e of the exact sequence (1) and the isomorphism $H^p(W, \Omega^{3-p}_W(Y, S_t)) \cong R'_{W,t}(K_W \otimes L^{p+1})$ is given by repeated cup product with e, it follows that we can identify the complex (3) with the complex

$$\bigwedge^2 A \otimes R'(K_W \otimes L) \to A \otimes R'(K_W \otimes L^2) \to R'(K_W \otimes L^3) \to 0.$$

The latter complex is exact if

(i) The complex

$$\bigwedge^2 A \otimes H^0(W, K_W \otimes L) \to A \otimes H^0(W, K_W \otimes L^2) \to H^0(W, K_W \otimes L^3) \to 0$$

is exact;

(ii) The map $A \otimes J'_{W,t}(K_W \otimes L^2) \to J'_{W,t}(K_W \otimes L^3)$ is surjective.

The first statement follows from [G1, Lemma 2.47]. For the second statement we consider the commutative diagram

$$\begin{array}{ccc}
A \otimes H^0(K_W \otimes \Sigma_{W,L}(-\log Y) \otimes L) & \xrightarrow{\mu} & H^0(K_W \otimes \Sigma_{W,L}(-\log Y) \otimes L^2) \\
\downarrow {\scriptstyle \cdot j^1(t)} & & \downarrow {\scriptstyle \cdot j^1(t)} \\
A \otimes J'_{W,t}(K_W \otimes L^2) & \xrightarrow{\nu} & J'_{W,t}(K_W \otimes L^3).
\end{array}$$

By [G1, Lemma 1.28] the map μ is surjective if L is sufficiently ample, hence ν is surjective because the vertical arrows are surjective by definition. $\square$

Corollary 1.6. *If L is sufficiently ample then*

(i) $R^a f_* \Omega^b_{W_A}(S_A, Y_A) = 0$ *if* $a + b \leq 4$ *and* $b \geq 2$;

(ii) $R^a f_* \Omega^b_{W_T}(S_T, Y_T) = 0$ *if* $a + b \leq 4$ *and* $b \geq 2$.

Proof: (i): Let $L^\bullet$ be the Leray filtration on $\Omega^b_{W_A}(S_A, Y_A)$. Consider the spectral sequence

$$E_1^{p,q} = R^{p+q} f_* \operatorname{Gr}^p_L \Omega^b_{W_A}(S_A, Y_A) \Rightarrow R^{p+q} f_* \Omega^b_{W_A}(S_A, Y_A).$$

We have $E_1^{p,q-b} \cong \bigwedge^p A^\vee \otimes \mathcal{H}^{b-p,p+q-b}_{U,V}$. The complex (2) can be identified with $E_1^{\bullet,3-b}$. By Lemma 1.4 we have $E_1^{p,q-b} = 0$ if $q \neq 3$, hence

$$\begin{array}{rcl} R^{k-b} f_* \Omega^b_{W_A}(S_A, Y_A) & = & 0 \quad \text{if } k \leq 2 \\ R^{3-b} f_* \Omega^b_{W_A}(S_A, Y_A) & \cong & E_2^{0,3-b} \\ R^{4-b} f_* \Omega^b_{W_A}(S_A, Y_A) & \cong & E_2^{1,3-b} \end{array}$$

and the assertion follows from Lemma 1.5.

(ii): Consider the chain of morphisms

$$T \xrightarrow{f_1} B \xleftarrow{g_1} \mathbb{P}H^0(W, L) \setminus \Delta \xrightarrow{f_2} \mathbb{P}H^0(W, L) \xleftarrow{g_2} H^0(W, L) \setminus \{0\} \xrightarrow{f_3} A.$$

As f_1, f_2, f_3 are smooth and g_1, g_2 are smooth and surjective, the assertion follows from (i) using [N, Lemma 2.2]. $\square$

Theorem 1.7. *If $Y \subset W$ is an ample divisor and if L is sufficiently ample, then* $H^4_{\mathcal{D}}(U_T, V_T, \mathbb{Q}(2)) = 0$.

Proof: We have an exact sequence

$$H^3(U_T, V_T) \to H^4_{\mathcal{D}}(U_T, V_T, \mathbb{Q}(2)) \to H^4(U_T, V_T) \oplus \mathbb{H}^4(\sigma_{\geq 2} \Omega^\bullet_{W_T}(S_T, Y_T)).$$

By Corollary 1.2 and Lemma 1.3 it suffices to show that

$$\mathbb{H}^4(\sigma_{\geq 2} \Omega^\bullet_{W_T}(S_T, Y_T)) = 0. \tag{4}$$

To this end, we use the Grothendieck spectral sequence

$$E_2^{p,q} = H^p(T, \mathbb{R}^q f_* \sigma_{\geq 2} \Omega^\bullet_{W_T}(S_T, Y_T)) \Rightarrow \mathbb{H}^{p+q}(\sigma_{\geq 2} \Omega^\bullet_{W_T}(S_T, Y_T)).$$

To verify the assertion (4) it suffices to show that

$$\mathbb{R}^q f_* \sigma_{\geq 2} \Omega^\bullet_{W_T}(S_T, Y_T) = 0 \quad \text{if } q \leq 4.$$

This assertion follows from Corollary 1.6 using the spectral sequence

$$E_1^{a,b} = R^a f_* \sigma_{\geq 2} \Omega^b_{W_T}(S_T, Y_T) \Rightarrow \mathbb{R}^{a+b} f_* \sigma_{\geq 2} \Omega^\bullet_{W_T}(S_T, Y_T).$$

□

Lemma 1.8. *If $Z_T \in \mathrm{CH}_0(S_T/T)$ is a relative zero cycle such that $Z(t)$ is rationally equivalent to zero for all $t \in T_{\mathbb{C}}$, then there exists a Zariski open subset $U \subset T$ such that the Deligne cycle class $\mathrm{cl}_{\mathcal{D}}(Z_U) \in H^4_{\mathcal{D}}(S_U, \mathbb{Q}(2))$ is zero.*

Proof: We use the methods of Bloch's proof of Mumford's theorem [B]. We may assume that T is irreducible. Let η be the generic point of T and let K be the function field of T. Choose an embedding of K in $\mathbb{C}$. The generic point η defines closed points η_K and $\eta_{\mathbb{C}}$ of T_K and $T_{\mathbb{C}}$. By assumption we have $[Z(\eta_K)] \in \ker(\mathrm{CH}^2(S_K) \to \mathrm{CH}^2(S_{\mathbb{C}}))$. The kernel of this map is a torsion group [loc.cit., Lemma 3], hence there exists $N \in \mathbb{N}$ such that $N[Z(\eta_K)] = 0$ in $\mathrm{CH}^2(S_K)$. As

$$\mathrm{CH}^2(S_K) = \varinjlim_{V \subset T \text{ open}} \mathrm{CH}^2(S_V)$$

there exists a Zariski open subset $U \subset T$ such that $N[Z_U] = 0$ in $\mathrm{CH}^2(S_U)$, hence $\mathrm{cl}_{\mathcal{D}}(Z_U) = 0$ in $H^4_{\mathcal{D}}(S_U, \mathbb{Q}(2))$. □

Define $\mathrm{CH}^1(Y)_0 = \{z \in \mathrm{CH}^1(Y)_{\mathbb{Q}} |\, \deg(z|_C) = 0\}$.

Proposition 1.9. *If $t \in T$ is very general and L is sufficiently ample, the restriction map $j_t^* : \mathrm{CH}^1(Y)_0 \to A_0(C_t)_{\mathbb{Q}}$ is injective.*

Proof: Set $\mathrm{Hdg}^1_{\mathrm{pr}}(Y)_{\mathbb{Q}} = H^{1,1}_{\mathrm{pr}}(Y) \cap H^2(Y, \mathbb{Q})$. We have an exact sequence

$$0 \to \mathrm{Pic}^0(Y)_{\mathbb{Q}} \to \mathrm{CH}^1(Y)_0 \to \mathrm{Hdg}^1_{\mathrm{pr}}(Y)_{\mathbb{Q}} \to 0.$$

Using the Lefschetz hyperplane theorem one shows that the restriction map $\mathrm{Pic}^0(Y) \to A_0(C_t)$ is injective, hence the kernel of $j_t^* : \mathrm{CH}^1(Y)_0 \to A_0(C_t)$ coincides with the kernel of the map

$$\psi : \mathrm{Hdg}^1_{\mathrm{pr}}(Y)_{\mathbb{Q}} \to J(C_t)_{\mathrm{var}} \otimes \mathbb{Q} \cong A_0(C_t)/\,\mathrm{Pic}^0(Y) \otimes \mathbb{Q}$$

obtained by lifing a primitive Hodge class to $\mathrm{CH}^1(Y)_0$ and restricting to C_t. Let $U' \subset \mathbb{P}H^0(Y, L|_Y)$ be the complement of the discriminant locus. Set $B' = U'/\operatorname{Aut}(Y)$ and let $\mathcal{J}$ be the Jacobian fibration associated to the universal family $f : C_{B'} \to B'$. Let $\nu_\xi \in H^0(B', \mathcal{J})$ be the normal function associated to ξ and let $\mathbb{V} = (R^1 f_* \mathbb{Q})_{\mathrm{var}}$ be the local system of variable cohomology. As in [K, Lemme 6.4.2] one shows that $\mathbb{V} \neq 0$ if L is sufficiently ample. We study the normal function ν_ξ by a technique due to Griffiths: using restriction to a Lefschetz pencil and results of N. Katz [loc. cit., Thms. 5.8.4 and 6.3] and Zucker [Z, Prop. 3.9] one shows that the map $\partial : \mathrm{Hdg}^1_{\mathrm{pr}}(Y)_\mathbb{Q} \to H^1(U', \mathbb{V})$ that associates to a normal function its cohomological invariant is injective if L is sufficiently ample, hence $\nu_\xi \in H^0(T, \mathcal{J})$ is not torsion if $\mathrm{cl}_Y(\xi) \in \mathrm{Hdg}^1_{\mathrm{pr}}(Y)_\mathbb{Q}$ is nonzero. There is a natural restriction map $r : B \to B'$ which is surjective if L is sufficiently ample. If $g : T \to B$ is a finite étale morphism, the morphism $h = r \circ g : T \to B'$ is dominant, hence $h^* \nu_\xi$ is not a torsion section of the induced Jacobian fibration over T. It follows that for very general $t \in T$ the map $\psi : \mathrm{Hdg}^1_{\mathrm{pr}}(Y)_\mathbb{Q} \to J(C_t)_{\mathrm{var}} \otimes \mathbb{Q}$ is injective, and the assertion follows. $\square$

2 Extension of Voisin's theorem

Let W be a smooth projective threefold and let $Y \subset W$ be a smooth ample divisor. Let L be an ample line bundle on W such that $H^0(W, L) \neq 0$ (this condition is satisfied if L is sufficiently ample). For $t \in H^0(W, L)$ we define $S_t = V(t)$, $C_t = Y \cap S_t$. Let G be the group of automorphisms of W that preserve Y and define

$$\Delta = \{[t] \in \mathbb{P}H^0(W, L) | C_t \text{ is singular}\}.$$

Set $B = (\mathbb{P}H^0(W, L) \setminus \Delta)/G$, $W_B = W \times B$, $Y_B = Y \times B$. Let $S_B \subset W_B$ be the universal family. Set $C_B = Y_B \cap S_B$, $U_B = W_B \setminus Y_B$ and $V_B = S_B \setminus C_B$.

Let $i_t : C_t \to S_t$ and $j_t : C_t \to Y$ be the inclusion morphisms. Consider the compositions

$$\begin{aligned} f_1 : \mathrm{CH}^1(Y) &\xrightarrow{i_*} \mathrm{CH}^2(W) \xrightarrow{\mathrm{cl}_{\mathcal{D},W}} H^4_{\mathcal{D}}(W, \mathbb{Q}(2)) \\ f_2 : \mathrm{CH}^1(Y) &\xrightarrow{j_t^*} \mathrm{CH}^1(C_t) \xrightarrow{\mathrm{cl}_{C_t}} H^2(C_t, \mathbb{Q}) \end{aligned}$$

and define

$$\mathrm{CH}^1(Y)_{\mathrm{var}} = \ker f_1, \quad \mathrm{CH}^1(Y)_0 = \ker f_2.$$

Let $g : H^4_{\mathcal{D}}(W, \mathbb{Q}(2)) \to H^4(S_t, \mathbb{Q})$ be the composition of the restriction map $H^4_{\mathcal{D}}(W, \mathbb{Q}(2)) \to H^4_{\mathcal{D}}(S_t, \mathbb{Q}(2))$ and the projection to $H^4(S_t, \mathbb{Q})$. The commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^1(Y) & \xrightarrow{f_1} & H^4_{\mathcal{D}}(W, \mathbb{Q}(2)) \\ \downarrow f_2 & & \downarrow g \\ H^2(C_t, \mathbb{Q}) & \xrightarrow{(i_t)_*} & H^4(S_t, \mathbb{Q}) \end{array}$$

shows that $\mathrm{CH}^1(Y)_{\mathrm{var}} \subseteq \mathrm{CH}^1(Y)_0$ (the lower horizontal map is an isomorphism by the Lefschetz hyperplane theorem). Hence $j_t^* \,\mathrm{CH}^1(Y)_{\mathrm{var}} \subseteq A_0(C_t)_{\mathbb{Q}}$.

Theorem 2.1. *If $t \in B$ is very general and if L is sufficiently ample, the kernel of the map*

$$(i_t)_* : A_0(C_t)_{\mathbb{Q}} \to A_0(S_t)_{\mathbb{Q}}$$

is contained in the image of the map

$$j_t^* : \mathrm{CH}^1(Y)_{\mathrm{var}} \to A_0(C_t)_{\mathbb{Q}}.$$

Proof: A standard argument shows that if $b \in B$ is very general and $z_0 \in ker(i_b)_*$ there exist a finite étale covering $f : T \to B$, a relative cycle $Z_T \in A_0(C_T/T)$ such that $Z(t) \in \ker(i_t)_*$ for all $t \in T_{\mathbb{C}}$ and a point $t_0 \in f^{-1}(b)$ such that $Z(t_0) = z_0$; cf. [V1, p. 85–86]. By Lemma 1.8 we may assume, after replacing T by a Zariski open subset, that $\mathrm{cl}_{\mathcal{D}}(Z_T) = 0$ in $H^4_{\mathcal{D}}(S_T, \mathbb{Q}(2))$. Consider the commutative diagram

$$\begin{array}{ccccc} H^3_{\mathcal{D}}(U_T, \mathbb{Q}(2)) & \to & H^3_{\mathcal{D}}(V_T, \mathbb{Q}(2)) & \to & H^4_{\mathcal{D}}(U_T, V_T, \mathbb{Q}(2)) \\ \downarrow r & & \downarrow & & \downarrow \\ H^2_{\mathcal{D}}(Y_T, \mathbb{Q}(1)) & \xrightarrow{k^*} & H^2_{\mathcal{D}}(C_T, \mathbb{Q}(1)) & \to & H^3_{\mathcal{D}}(Y_T, C_T, \mathbb{Q}(1)) \\ \downarrow & & \downarrow i_* & & \downarrow \\ H^4_{\mathcal{D}}(W_T, \mathbb{Q}(2)) & \to & H^4_{\mathcal{D}}(S_T, \mathbb{Q}(2)) & \to & H^5_{\mathcal{D}}(W_T, S_T, \mathbb{Q}(2)). \end{array}$$

As $\mathrm{cl}_{\mathcal{D}, C_T}(Z_T) \in \ker i_*$ and $H^4_{\mathcal{D}}(U_T, V_T, \mathbb{Q}(2)) = 0$ by Theorem 1.7, we can lift $\mathrm{cl}_{\mathcal{D}, C_T}(Z_T)$ to an element $\tilde{\xi} \in H^3_{\mathcal{D}}(U_T, \mathbb{Q}(2))$. The element $\xi = r(\tilde{\xi})$ belongs to

$$H^2_{\mathcal{D}}(Y_T, \mathbb{Q}(1))_{\mathrm{var}} = \ker(H^2_{\mathcal{D}}(Y_T, \mathbb{Q}(1)) \to H^4_{\mathcal{D}}(W_T, \mathbb{Q}(2))).$$

Let $a_{t_0} : Y \cong Y \times \{t_0\} \to Y_T$ and $b_{t_0} : C_{t_0} \to C_T$ be the inclusion morphisms. Consider the commutative diagram

$$\begin{array}{ccc} H^2_{\mathcal{D}}(Y_T, \mathbb{Q}(1))_{\mathrm{var}} & \xrightarrow{k^*} & H^2_{\mathcal{D}}(C_T, \mathbb{Q}(1)) \\ \Big\downarrow a^*_{t_0} & & \Big\downarrow b^*_{t_0} \\ H^2_{\mathcal{D}}(Y, \mathbb{Q}(1))_{\mathrm{var}} & \xrightarrow{j^*_{t_0}} & H^2_{\mathcal{D}}(C_{t_0}, \mathbb{Q}(1)). \end{array}$$

Set $\xi_0 = a^*_{t_0}\xi \in H^2_{\mathcal{D}}(Y, \mathbb{Q}(1))_{\mathrm{var}} \cong \mathrm{CH}^1(Y)_{\mathrm{var}}$. By construction we have $j^*_{t_0}\xi_0 = [Z(t_0)] = z_0 \in A_0(C_{t_0})$, hence $z_0 \in j^*_{t_0} \mathrm{CH}^1(Y)_{\mathrm{var}}$. $\square$

Remark 2.2.

(i) If the Deligne cycle class map on $\mathrm{CH}^2(W)_{\mathbb{Q}}$ is injective, it follows that $\mathrm{CH}^1(Y)_{\mathrm{var}} = \ker(\mathrm{CH}^1(Y)_{\mathbb{Q}} \to \mathrm{CH}^2(W)_{\mathbb{Q}})$ and $j^*_t \mathrm{CH}^1(Y)_{\mathrm{var}} \subseteq \ker(i_t)_*$ for all $t \in T$, with equality if $t \in B$ is very general. By [BS, Thm. 1 (i)], the Deligne cycle class map is injective if W is a Fano threefold (it is even an isomorphism). In general the Deligne cycle class map on $\mathrm{CH}^2(W)_{\mathbb{Q}}$ is not expected to be injective, and the inclusion $j^*_t \mathrm{CH}^1(Y)_{\mathrm{var}} \subseteq \ker(i_t)_*$ may fail to hold.

(ii) If $W = \mathbb{P}^3$, $L = \mathcal{O}_{\mathbb{P}}(d)$ and $Y \subset \mathbb{P}^3$ is a smooth surface of degree e, regularity computations show that Theorem 2.1 holds if $d \geq \max(5, 3e-3)$.

(iii) It is possible to prove Theorem 1.7 without the assumption that $Y \subset W$ is ample. If $Y \subset W$ is an arbitrary smooth subvariety of codimension one, Lemma 1.4 only gives $\mathcal{H}^q_{U,V} = 0$ if $q < 3$. In this case one has to use more sophisticated techniques to analyse the behaviour of the spectral sequence $E^{p,q}_r$ that appears in the proof of Corollary 1.6 (i).

It is possible to generalise Theorem 2.1 to the case of complete intersections (One reduces to the codimension one case by replacing W by a suitable projective bundle P and S_t by a global section of the tautological bundle $\mathcal{O}_P(1)$.) As an example, we mention the following result that generalises Voisin's theorem to the case of complete intersections:

Theorem 2.3. *Let $S = V(d_0, \dots, d_r) \subset \mathbb{P}^{r+3}$ be a complete intersection surface and let $H \subset \mathbb{P}^{r+3}$ be a hyperplane. Set $C = S \cap H$. If S is very general, the kernel of $i_* : A_0(C) \to A_0(S)$ is the torsion subgroup of $A_0(C)$ unless* *(i)* $r = 0$ *and* $d_0 \leq 4$*;*

(ii) $r = 1$ *and* $(d_0, d_1) = (d, 2)$*,* $d \geq 2$*;*

(iii) $r = 2$ *and* $(d_0, d_1, d_2) = (2, 2, 2)$*.*

The degree bounds in this example are sharp. For case (i) this has been shown by Voisin; see [V1, Remarque 2.3]. Case (iii) is treated using the same method. In case (ii) we consider surfaces S_τ, $\tau = (t, u)$, that are complete intersections of a smooth quadric $Q_t \subset \mathbb{P}^4$ and a hypersurface X_u of degree d. Choose $t_0 \in H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}}(2))$ and set $Y = Q_{t_0} \cap H$, $W = Q_{t_0}$. Let $\xi = L_1 - L_2$ be the difference of two lines from different rulings on Y. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^1(Y) & \rightarrow & \mathrm{CH}^2(W) \\ \Big\downarrow{\scriptstyle \cap X_u} & & \Big\downarrow{\scriptstyle \cap X_u} \\ \mathrm{CH}^1(C_\tau) & \xrightarrow{i_*} & \mathrm{CH}^2(S_\tau). \end{array}$$

As the cycle class map induces an isomorphism $\mathrm{CH}^2(Q_t) \cong H^4(Q_t, \mathbb{Z}) = \mathbb{Z}$ we have $[\xi] \in \mathrm{CH}^1(Y)_{\mathrm{var}}$, hence $[\xi \cap X] \in \ker i_*$. Let $U \subset \mathbb{P}H^0(Y, \mathcal{O}_Y(d))$ be the complement of the discriminant locus. If $d \geq 2$ then $H^1_{\mathrm{var}}(C_\tau, \mathbb{Q}) \neq 0$, and the proof of Proposition 1.9 shows that the map $\mathrm{CH}^1(Y)_{\mathrm{var}} \to A_0(C_\tau)_{\mathbb{Q}}$ is injective if $u \in U$ is very general. Set $E = \mathcal{O}_{\mathbb{P}^4}(2) \oplus \mathcal{O}_{\mathbb{P}^4}(d)$, and let $\Delta \subset \mathbb{P}H^0(\mathbb{P}^4, E)$ be the discriminant locus. Put $G = \{g \in \mathrm{PGL}(5) | g(H) = H\}$ and $B = (\mathbb{P}H^0(\mathbb{P}^4, E) \setminus \Delta)/G$. From the previous result and the transitivity of the PGL–action on the space of smooth quadrics one deduces the existence of a countable number of proper Zariski closed subsets $\Sigma_n \subset B$ such that if $d \geq 2$ and $\tau = [(t, u)] \notin \cup_{n \in \mathbb{N}} \Sigma_n$, the restriction map $j^* : \mathrm{CH}^1(Y_t)_{\mathrm{var}} \to A_0(C_\tau)_{\mathbb{Q}}$ is injective. It follows that $j^*\xi$ is a nonzero element of $\ker(i_\tau)_*$.

Remark 2.4. The behaviour of $\ker i_*$ is related to the problem of finding a criterion for a zero cycle on a smooth surface S to be rationally equivalent to zero. Let $T(S)$ be the kernel of the Albanese map. M. Green [G3] has defined a 'higher Abel–Jacobi map' $\psi^2_2 : T(S) \to J^2_2(S)$. Let $C \subset S$ be a smooth curve. The kernel of the map $i_* : A_0(C) \to A_0(S)$ is contained in the kernel of the composed map $f = \psi^2_2 \circ i_* : J(C)_{\mathrm{var}} \to J^2_2(S)$. Voisin [V2] has shown that the map ψ^2_2 may fail to be injective (see [BDPR] for another counterexample), hence the inclusion $\ker i_* \subseteq \ker f$ can be strict.

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