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Chapter 0

Introduction

Consider the following classical problem in complex analysis: let $\{p_i\}_{i\in I}$ be a discrete set of points in the complex plane, and let $\{a_i\}_{i\in I}$ be a set of integers. Determine whether there exists a meromorphic function f such that f has a zero of order a_i at p_i if $a_i \geq 0$, a pole of order $|a_i|$ at p_i if $a_i < 0$ and no zeroes or poles outside the set $\{p_i\}$. Thus the problem is to find a meromorphic function f such that the principal divisor

$$\operatorname{div}(f) = \sum_{p\in\mathbb{C}} \operatorname{ord}_p(f)$$

equals $\sum_{i\in I} a_i p_i$. It follows from a theorem of Weierstrass that this is always possible; see e.g. [Ah, Chapter 5, Thm. 7]. For the modern sheaf–theoretic approach see [Gu, pp. 121–126].

When we replace $\mathbb{C}$ by a compact complex manifold C of dimension one (a compact Riemann surface), the question becomes harder. In this case we consider a finite set of points $\{p_1,\ldots,p_k\}$ with multiplicities $\{a_1,\ldots,a_k\}$, i.e., a divisor $D = \sum_{i=1}^k a_i p_i$. The divisor D is called effective if $a_i \geq 0$ for $i = 1,\ldots,k$; its degree is $\deg(D) = \sum_{i=1}^k a_i$. A divisor is called *linearly equivalent to zero* if there exists a meromorphic function f on C such that $D = \operatorname{div}(f)$. As $\deg(\operatorname{div}(f)) = 0$, a divisor D can be linearly equivalent to zero only if $\deg(D) = 0$. If $C = \mathbb{P}^1$, this condition is also sufficient.

For divisors on Riemann surfaces of arbitrary genus, one can determine whether a divisor is linearly equivalent to zero by attaching a new invariant, the Abel–Jacobi invariant, to divisors of degree zero. To obtain this invariant, choose a topological 1–chain γ whose boundary is D. Integration over γ defines a linear functional $f_\gamma \in H^0(C,\Omega^1_C)^\vee$ on the space of global holomorphic 1–forms on C, and if we divide out by the lattice $H_1(C,\mathbb{Z})$ we obtain a well–defined element $\psi_C(D) = [f_\gamma]$ in the complex torus $J(C) = H^0(C,\Omega^1_C)^\vee/H_1(C,\mathbb{Z})$. Let $\operatorname{Div}^0(C)$ be the group of divisors of degree zero.

The map

$$\psi_C : \mathrm{Div}^0(C) \to J(C)$$

is called the Abel–Jacobi map. The solution to our original problem is given by the following theorem of Abel:

Theorem 0.0.1. *Let $D = \sum_{i=1}^{k} a_i p_i$ be a divisor on a compact Riemann surface C. There exists a meromorphic function f such that $D = \mathrm{div}(f)$ if and only if*

(i) $\deg(D) = 0$

(ii) $\psi_C(D) = 0$.

It is known that every compact Riemann surface C of genus g is a smooth projective algebraic variety. The kth symmetric product $C^{(k)}$ is defined as the quotient of the k-fold product $C \times \ldots \times C$ by the action of the symmetric group S_k; it is a smooth variety that parametrizes effective divisors of degree k. If we fix a base point $p_0 \in C$, we obtain a map

$$\Phi_k : C^{(k)} \to J(C)$$

that sends $p_1 + \ldots + p_k$ to the linear functional $\Phi_k(p_1 + \ldots + p_k)$ defined by

$$\Phi_k(p_1 + \ldots + p_k)(\omega) = \sum_{i=1}^{k} \int_{p_0}^{p_i} \omega.$$

This map is generically injective if $k < g$; it is surjective if $k \geq g$ (Jacobi inversion theorem). The complex torus $J(C)$ is called the Jacobian of C. It is an abelian variety whose geometry is closely related to the geometry of linear systems on C (see [ACGH]).

We turn to higher dimensional varieties. An *algebraic cycle* of codimension p is a finite linear combination, with integral coefficients, of irreducible subvarieties of codimension p in X. The group of algebraic cycles of codimension p is denoted by $Z^p(X)$. An algebraic cycle $Z = \sum n_i Z_i$ is called effective if $n_i \geq 0$ for all i. The two invariants that we considered before (the degree and the Abel–Jacobi invariant) generalize to the higher dimensional case. The degree is replaced by the cycle class map

$$\mathrm{cl}_X : Z^p(X) \to H^{2p}(X, \mathbb{Z})$$

that sends $Z = \sum n_i Z_i$ to $\sum n_i [Z_i]$, where $[Z_i]$ is the fundamental class of Z_i. An algebraic cycle $Z \in Z^p(X)$ is called *homologically equivalent to zero* (notation $Z \sim_{\mathrm{hom}} 0$) if $\mathrm{cl}_X(Z) = 0$. Set

$$Z^p_{\mathrm{hom}}(X) = \ker \mathrm{cl}_X = \{Z \in Z^p(X) : Z \sim_{\mathrm{hom}} 0\}.$$

The generalization of linear equivalence is *rational equivalence*: let $W \subset X$ be a subvariety of codimension $p-1$, and let $f \in k(W)^*$ be a rational function on W. We define

$$\operatorname{div}(f) = \sum_{V \subset W} \operatorname{ord}_V(f).V,$$

where the sum is taken over all subvarieties $V \subset W$ of codimension one. An algebraic cycle $Z \in Z^p(X)$ is called rationally equivalent to zero if there exists a finite collection $\{(W_i, f_i)\}$ of subvarieties W_i of codimension $p-1$ and rational functions $f_i \in k(W_i)^*$ such that

$$Z = \sum_i \operatorname{div}(f_i).$$

Two algebraic cycles Z and Z' are called *algebraically equivalent* if there exists a family of cycles $\mathcal{W} \subset X \times T$ parametrized by an irreducible variety T and two points t_0, t_1 in T such that $Z = \mathcal{W}_{t_0} = \mathcal{W} \cap X \times \{t_0\}$ and $Z' = \mathcal{W}_{t_1} = \mathcal{W} \cap X \times \{t_1\}$.

Remark 0.0.2. In the definition of algebraic equivalence we may assume that T is a smooth, irreducible curve. If T is rational, we recover the notion of rational equivalence.

Set

$$\begin{aligned} Z^p_{\text{rat}}(X) &= \{Z \in Z^p(X) : Z \sim_{\text{rat}} 0\} \\ Z^p_{\text{alg}}(X) &= \{Z \in Z^p(X) : Z \sim_{\text{alg}} 0\}. \end{aligned}$$

We have inclusions

$$Z^p_{\text{rat}}(X) \subseteq Z^p_{\text{alg}}(X) \subseteq Z^p_{\text{hom}}(X) \subset Z^p(X).$$

The quotients are denoted by

$$\begin{aligned} \operatorname{CH}^p(X) &= Z^p(X)/Z^p_{\text{rat}}(X) \\ \operatorname{CH}^p_{\text{alg}}(X) &= Z^p_{\text{alg}}(X)/Z^p_{\text{rat}}(X) \\ \operatorname{CH}^p_{\text{hom}}(X) &= Z^p_{\text{hom}}(X)/Z^p_{\text{rat}}(X) \\ \operatorname{Griff}^p(X) &= Z^p_{\text{hom}}(X)/Z^p_{\text{alg}}(X). \end{aligned}$$

The group $\operatorname{CH}^p(X)$ is called the Chow group of codimension p cycles on X, and $\operatorname{Griff}^p(X)$ is called the Griffiths group of codimension p cycles.

To define the Abel–Jacobi map on X, we recall some basic facts from Hodge theory. Let $A^k(X)$ be the set of complex–valued C^∞ k–forms on X. It admits a decomposition

$$A^k(X) = \bigoplus\nolimits_{p+q=k} A^{p,q}(X)$$

where $A^{p,q}(X)$ denotes the set of C^∞ k–forms of type (p,q), i.e., the k–forms that can be expressed in local coordinates as

$$\omega = \sum_{|I|=p,|J|=q} f_{I,J} dz_I \wedge d\overline{z}_J.$$

Set

$$F^pA^k(X) = \bigoplus\nolimits_{r\geq p} A^{r,k-r}(X),$$

and let $A^k_c(X)$ be the set of closed k–forms. De Rham's theorem shows that $H^k(X,\mathbb{C}) \cong A^k_c(X)/dA^{k-1}(X)$. By the Hodge theorem, the decomposition of $A^k(X)$ into forms of type (p,q) induces a decomposition

$$H^k(X,\mathbb{C}) \cong \bigoplus\nolimits_{p+q=k} H^{p,q}(X),$$

the Hodge decomposition. The associated Hodge filtration is

$$H^k(X,\mathbb{C}) = F^0H^k(X,\mathbb{C}) \supset \ldots \supset F^kH^k(X,\mathbb{C}) \supset F^{k+1}H^k(X,\mathbb{C}) = 0,$$

where

$$F^pH^k(X,\mathbb{C}) \cong \bigoplus\nolimits_{r\geq p} H^{r,k-r}(X).$$

One can summarize this by saying that $H^k(X,\mathbb{Z})$ carries a Hodge structure of weight k; see [G4, Lecture 1] for the definition.

If $\mathrm{cl}_X(Z) = 0$, there exists a topological $(2n-2p+1)$–chain γ such that $\partial\gamma = Z$. Integration over γ defines a linear functional

$$f_\gamma \in F^{n-p+1}A^{2n-2p+1}(X)^\vee.$$

If $\omega \in F^{n-p+1}A^{2n-2p+1}(X)$ is exact, there exists a form $\eta \in F^{n-p+1}A^{2n-2p}(X)$ such that $d\eta = \omega$ by the principle of two types [loc.cit.]. By the theorem of Stokes the functional f_γ descends to a functional on $F^{n-p+1}H^{2n-2p+1}(X,\mathbb{C})^\vee$. The image of $H_{2n-2p+1}(X,\mathbb{Z})$ in $F^{n-p+1}H^{2n-2p+1}(X,\mathbb{C})^\vee$ is a lattice Λ, and the image of f_γ in the complex torus

$$J^p(X) = F^{n-p+1}H^{2n-2p+1}(X,\mathbb{C})^\vee/\Lambda$$

is well–defined; it is called the Abel–Jacobi invariant $\psi_X(Z)$ of Z. As the annihilator of $F^{n-p+1}H^{2n-2p+1}(X,\mathbb{C})$ under the cup product pairing

$$H^{2n-2p+1}(X,\mathbb{C}) \times H^{2p-1}(X,\mathbb{C}) \to \mathbb{C}$$

is $F^pH^{2p-1}(X,\mathbb{C})$, we can identify $J^p(X)$ with the quotient

$$H^{2p-1}(X,\mathbb{C})/F^pH^{2p-1}(X,\mathbb{C}) + \mathrm{im}\, H^{2p-1}(X,\mathbb{Z}).$$

The complex torus $J^p(X)$ is called the pth intermediate Jacobian. If $p = 1$ we obtain the Picard variety

$$\mathrm{Pic}^0(X) = H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z});$$

if $p = n$ we obtain the Albanese variety

$$\mathrm{Alb}(X) = H^0(X, \Omega^1_X)^\vee / H^1(X, \mathbb{Z}).$$

Thus $J^p(X)$ is an abelian variety if $p = 1$ or $p = n$. In general, $J^p(X)$ is not an abelian variety if $1 < p < n$. Since every map from $\mathbb{P}^1$ to a complex torus is constant, it follows that $\psi_X(Z^p_{\mathrm{rat}}(X)) = 0$; hence there is an induced map

$$\psi_X : \mathrm{CH}^p_{\mathrm{hom}}(X) \to J^p(X).$$

In the case of divisors (codimension one cycles), homological and algebraic equivalence coincide. Hence the image of

$$\mathrm{cl}_X : \mathrm{CH}^1(X) \to H^2(X, \mathbb{Z}),$$

which is a finitely generated abelian group, is identified with the Néron–Severi group of divisors modulo algebraic equivalence. The Abel–Jacobi map induces an isomorphism

$$\psi_X : \mathrm{CH}^1_{\mathrm{hom}}(X) = \mathrm{CH}^1_{\mathrm{alg}}(X) \to \mathrm{Pic}^0(X);$$

see for instance [V4, Lecture 1]. Thus the invariants cl_X and ψ_X determine whether a divisor is rationally equivalent to zero, as in the case of curves.

For algebraic cycles of codimension $p \geq 2$ the situation is more complicated. Even the case of zero–cycles on surfaces presents considerable difficulties, as is illustrated by Mumford's famous result on 'infinite dimensionality' of the kernel of the Albanese map

$$\psi_X : \mathrm{CH}^2_{\mathrm{hom}}(X) \to \mathrm{Alb}(X)$$

for a surface X with $p_g(X) \neq 0$. Bloch has conjectured that there exists a decreasing filtration

$$\mathrm{CH}^p(X)_\mathbb{Q} = F^0\,\mathrm{CH}^p(X)_\mathbb{Q} \supset \ldots \supset F^p\,\mathrm{CH}^p(X)_\mathbb{Q} \supset F^{p+1}\,\mathrm{CH}^p(X)_\mathbb{Q} = 0$$

whose graded pieces should admit maps

$$\psi_k : F^k\,\mathrm{CH}^p(X)_\mathbb{Q} \to J^p_k(X)$$

to Hodge–theoretic objects $J_k^p(X)$ that are associated to $H^{2p-k}(X,\mathbb{C})$. The filtration $F^\bullet$ should be inductively defined as $F^{k+1} = \ker\psi_k \subset F^k$; for instance, we should have

$$\begin{aligned} F^1\,\mathrm{CH}^p(X)_\mathbb{Q} &= \mathrm{CH}^p_{\mathrm{hom}}(X)_\mathbb{Q} \\ F^2\,\mathrm{CH}^p(X)_\mathbb{Q} &= \ker\psi_{X,\mathbb{Q}}. \end{aligned}$$

If this program could be carried out, it would follow that $Z \in Z^p_{\mathrm{rat}}(X)$ if and only if $\psi_0(Z) = \psi_1(Z) = \ldots = \psi_p(Z) = 0$. There have been several proposals for the filtration $F^\bullet$, due to Beilinson, Murre and S. Saito; see the survey article [J] for a detailed discussion. Recently, Green [G5] has proposed a definition for the maps ψ_k.

So far, the only known invariants associated to algebraic cycles are the maps $\psi_0 = \mathrm{cl}_X$ and $\psi_1 = \psi_X$. A natural problem is to study their image. It is not difficult to show that the image of the rational cycle class map $\mathrm{cl}_{X,\mathbb{Q}}$ is contained in the set of rational Hodge classes

$$\mathrm{Hdg}^p(X)_\mathbb{Q} = \{\alpha \in H^{2p}(X,\mathbb{Q}) : j(\alpha) \in H^{p,p}(X,\mathbb{C})\},$$

where $j : H^{2p}(X,\mathbb{Q}) \to H^{2p}(X,\mathbb{C})$ is the natural inclusion map given by extension of scalars. The Hodge conjecture predicts that $\mathrm{im}\,\mathrm{cl}_{X,\mathbb{Q}} = \mathrm{Hdg}^p(X)_\mathbb{Q}$.

The Hodge conjecture holds for divisors (Lefschetz (1,1) theorem) and for 1–dimensional cycles, but little is known about it in general. If $\{X_t\}_{t\in T}$ is a family of hypersurface sections of a smooth projective variety Y, then there exists (under a mild hypothesis) a countable union $\{T_\alpha\}_{\alpha\in A}$ of proper analytic subsets of T such that the image of the rational cycle class map on X_t is determined by the image of the cycle class map on Y if $t \notin \cup_\alpha T_\alpha$ (such a point $t \in T$ is called *very general*). Thus if the image of cl_Y is known, we can describe the image of the cycle class map cl_{X_t} for very general $t \in T$. Let $i : X \to Y$ be the inclusion map, and define

$$H^{2m}_{\mathrm{var}}(X) = \ker i_* : H^{2m}(X) \to H^{2m+2}(Y).$$

Theorem 0.0.3. (Lefschetz) *Let Y be a smooth projective variety of dimension $2m+1$, and let $X = V(d) \subset Y$ be a degree d hypersurface section. If X is very general and $H^{p,2m-p}_{\mathrm{var}}(X) \neq 0$ for some $p < m$, then the image of*

$$\mathrm{cl}_{X,\mathbb{Q}} : \mathrm{CH}^m(X)_\mathbb{Q} \to H^{2m}(X,\mathbb{Q})$$

coincides with the image of

$$i^*\circ\mathrm{cl}_{Y,\mathbb{Q}} : \mathrm{CH}^m(Y)_\mathbb{Q} \to H^{2m}(X,\mathbb{Q}).$$

This result is usually stated only for $Y = \mathbb{P}^{2m+1}$. In that case, the conclusion of the Theorem even holds with coefficients in $\mathbb{Z}$; see [H1, Thm. 3.4], or [Shi]. The proof in the general case is exactly the same, and is based on the irreducibility of the monodromy action on $H^{2m}_{\mathrm{var}}(X, \mathbb{Q})$. The same argument can be applied for complete intersections (see [DK, Exposé XIX], again for the case that Y is a projective space). If we take $Y = \mathbb{P}^3$, then the condition of the theorem is satisfied for $d \geq 4$. Using the exponential sequence, one obtains the Noether–Lefschetz theorem:

Corollary 0.0.4. *If $X \subset \mathbb{P}^3$ is a very general surface of degree $d \geq 4$, then $\mathrm{CH}^1(X) \cong \mathbb{Z}$.*

Since the linear system $|\mathcal{O}_X(k)|$ cut out by hypersurfaces of degree k is complete, it even follows that every curve $C \subset X$ is the complete intersection of X with a surface in $\mathbb{P}^3$. In [GH2] Griffiths and Harris tried to generalize the Noether–Lefschetz theorem to the case of curves contained in hypersurfaces of degree $d \geq 6$ in $\mathbb{P}^4$ (a hypersurface of degree $d \leq 5$ in $\mathbb{P}^4$ always contains lines). Voisin [V1] showed that the strongest possible statement (every curve on X is a complete intersection) is false. It is not known whether $\mathrm{CH}^2(X) \cong \mathbb{Z}$. Instead, one can try to prove the weaker assertion $\operatorname{im} \psi_X = 0$ by establishing an analogue of Theorem 0.0.3 for the image of the Abel–Jacobi map.

Let X be a smooth projective variety. The image

$$J^m_{\mathrm{alg}}(X) = \psi_X(\mathrm{CH}^m_{\mathrm{alg}}(X))$$

of the Chow group of cycles that are algebraically equivalent to zero is an abelian subvariety of $J^m(X)$ associated to a sub Hodge structure contained in $H^{m-1,m}(X) + H^{m,m-1}(X)$. Let $J^m_a(X)$ be the abelian subvariety of $J^m(X)$ associated to the maximal $\mathbb{Q}$–sub Hodge structure contained in $H^{m-1,m}(X) + H^{m,m-1}(X)$. The (corrected) generalized Hodge conjecture predicts that $J^m_{\mathrm{alg}}(X)$ coincides with $J^m_a(X)$. Let Y be a smooth projective variety of dimension $2m$, and let $X = V(d) \subset Y$ be a very general hypersurface section. In this case we can describe $J^m_{\mathrm{alg}}(X)$ in terms of $J^m_{\mathrm{alg}}(Y)$, provided that d is sufficiently large. The proof is analogous to the proof of Theorem 0.0.3 and can be found in [H1] or [Shi] (again they consider the case $Y = \mathbb{P}^{2m}$, but the proof in the general case is the same).

Theorem 0.0.5. (Griffiths) *Let Y be a smooth projective variety of dimension $2m$, and let $X = V(d) \subset Y$ be a hypersurface section of degree d. If X is very general and $H^{p,2m-p-1}_{\mathrm{var}}(X) \neq 0$ for some $p < m-1$, then the image of*

$$\psi_{X,\mathbb{Q}} : \mathrm{CH}^m_{\mathrm{alg}}(X)_{\mathbb{Q}} \to J^m(X)_{\mathbb{Q}}$$

coincides with the image of

$$i^*\circ\psi_{Y,\mathbb{Q}} : \mathrm{CH}^m_{\mathrm{alg}}(Y)_{\mathbb{Q}} \to J^m(X)_{\mathbb{Q}}.$$

For instance, if $X = V(d) \subset \mathbb{P}^4$ is a very general threefold of degree $d \geq 5$ then $J^2_{\mathrm{alg}}(X) = 0$. We are thus left with the problem of determining the image of the Griffiths group $\mathrm{Griff}^m(X)$ under the induced map

$$\overline{\psi}_X : \mathrm{Griff}^m(X) \to J^m(X)/J^m_{\mathrm{alg}}(X).$$

As homological and algebraic equivalence coincide for zero–cycles and divisors, we consider curves on threefolds. For curves on Fano threefolds, i.e., threefolds whose anticanonical bundle is ample, homological and algebraic equivalence also coincide; see [BlS]. Griffiths [Gr2] gave examples of quintic threefolds containing two lines whose difference is not algebraically equivalent to zero. As the Chow scheme parametrizing effective cycles on X has countably many components, the Griffiths group and its image under $\overline{\psi}_X$ are countable groups. Clemens has shown that the Griffiths group of a very general quintic threefold is not finitely generated [C1]. The behaviour of the group $\overline{\psi}_X(\mathrm{Griff}^m(X))$ is mysterious. Using infinitesimal methods we shall see that it is possible to describe this group in terms of the image of cl_Y and ψ_Y for very general hypersurface sections $X = V(d) \subset Y$ of sufficiently large degree.

The use of infinitesimal methods in Hodge theory is motivated by the observation that an abstract Hodge structure of weight $k \geq 2$ (with $h^{2,0} > 1$ if $k = 2$) is generally not associated to a geometric object, due to the Griffiths transversality relation. To circumvent this problem one can consider not only the Hodge structure itself, but also its infinitesimal variation. The theory of infinitesimal variations of Hodge structures was introduced in [CGGH]; see also [PS]. A short exposition of the underlying ideas can be found in [Ha]. In brief, an infinitesimal variation of Hodge structure consists of a polarized Hodge structure $(H_{\mathbb{Z}}, F^{\bullet}, Q)$ of weight k, a vector space T_0 and a linear map

$$\delta = \textstyle\bigoplus_p \delta_p : T_0 \to \bigoplus_{p=1}^k \mathrm{Hom}(H^{p,k-p}, H^{p-1,k-p+1}).$$

subject to two additional conditions (see [CGGH] or [PS] for details). In the geometric situation, T_0 is the tangent space at 0 to the parameter space of a family of smooth projective varieties and the maps

$$\delta_p(v) : H^{p,q}(X_0) \to H^{p-1,q+1}(X_0)$$

are given by cup product with the Kodaira–Spencer class $\rho(v) \in H^1(X_0, T_{X_0})$, followed by contraction.

There is a lot of multilinear algebra related to infinitesimal variations of Hodge structure. For instance, we can perform the following construction: choose a basis $\{v_1, \ldots, v_k\}$ for $T_0^\vee$ and consider the complex $\bigwedge^\bullet T_0^\vee \otimes H$ with maps $d_q : \bigwedge^q T_0^\vee \otimes H \to \bigwedge^{q+1} T_0^\vee \otimes H$ given by

$$d_q(v_{i_1} \wedge \ldots \wedge v_{i_q} \otimes \alpha) = \sum_{i=1}^{k} v_i \wedge v_{i_1} \wedge \ldots \wedge v_{i_q} \otimes \langle v_i, \nabla(\alpha) \rangle,$$

where ∇ is the Gauss–Manin connection. This complex is exact, but the subcomplexes

$$F^p(\textstyle\bigwedge^\bullet T_0^\vee \otimes H) : F^p \to T_0^\vee \otimes F^{p-1} \to \textstyle\bigwedge^2 T_0^\vee \otimes F^{p-2} \to \cdots$$

need no longer be exact.

Let Y be a smooth projective variety of dimension $n+1$, and let $X = V(d) \subset Y$ be a smooth hypersurface section. Set $H = H^n_{\mathrm{var}}(X)$. As is shown in [G2], the Koszul cohomology groups $H^q(\mathrm{Gr}^p_F(\bigwedge^\bullet T_0^\vee \otimes H))$ carry a lot of geometric information. For instance, if $n = 2m$ the cohomology group $H^0(F^m(\bigwedge^\bullet T_0^\vee \otimes H))$ is related to an infinitesimal version of Theorem 0.0.3; see [CGGH, §3a] or [V4, Lecture 3].

We shall briefly explain how the cohomology group $H^1(F^m(\bigwedge^\bullet T_0^\vee \otimes H))$ is related to the behaviour of the Abel–Jacobi map on X if $n = 2m-1$. If X is very general, homologically trivial cycles on X can be 'spread out' to relative cycles over the universal family of smooth degree d hypersurfaces in Y; see Chapter 2. As a relative cycle obtained in this way restricts to homologically trivial cycles on the fibers, we can apply the Abel–Jacobi map on the fibers to obtain a global holomorphic section ν of the family $\mathcal{J}^m$ of intermediate Jacobians. This global section satisfies a horizontality property and is called a normal function; it has an infinitesimal invariant $\delta\nu$, whose value at 0 is $\delta\nu(0) \in H^1(F^m(\bigwedge^\bullet T_0^\vee \otimes H))$. If $\delta\nu = 0$, then ν is locally constant.

The primitive cohomology group $H^{m,m}_{\mathrm{pr}}(Y)$ is defined as the kernel of the map $H^{m,m}(Y) \to H^{m+1,m+1}(Y)$ given by cup product with the first Chern class of $\mathcal{O}_Y(1)$. If d is sufficiently large, one can show that the group $H^1(F^m(\bigwedge^\bullet T_0^\vee \otimes H))$ is isomorphic to $H^{m,m}_{\mathrm{pr}}(Y)$. This leads to a strengthening of Theorem 0.0.5, which is most conveniently phrased in terms of Deligne cohomology. Let $D^\bullet_X(m)$ be the truncated De Rham complex

$$0 \to \mathbb{Q} \to \mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^{m-1}_X \to 0.$$

The rational Deligne cohomology group $H^{2m}_{\mathcal{D}}(X, \mathbb{Q})$ is defined by

$$H^{2m}_{\mathcal{D}}(X, \mathbb{Q}) = \mathbb{H}^{2m}(D^\bullet_X(m)).$$

By construction, $H^{2m}_{\mathcal{D}}(X,\mathbb{Q})$ fits into an exact sequence

$$0 \to J^m(X)_{\mathbb{Q}} \to H^{2m}_{\mathcal{D}}(X,\mathbb{Q}) \to \mathrm{Hdg}^m(X)_{\mathbb{Q}} \to 0.$$

There are Deligne cycle class maps

$$\mathrm{cl}_{\mathcal{D},X} : \mathrm{CH}^m(X) \to H^{2m}_{\mathcal{D}}(X,\mathbb{Q})$$

that are compatible with the maps cl_X and ψ_X (see [EV] or [EZ]). The following result of Green and Müller–Stach [GM] gives an almost complete description of the image of ψ_X in terms of the geometry of Y:

Theorem 0.0.6. *Let Y be a smooth projective variety of dimension $2m$, and let $X = V(d) \subset Y$ be a very general hypersurface section of degree d with inclusion map $i : X \to Y$. If d is sufficiently large, then the image of*

$$\mathrm{CH}^m(X)_{\mathbb{Q}} \to H^{2m}_{\mathcal{D}}(X,\mathbb{Q}) \to H^{2m}_{\mathcal{D}}(X,\mathbb{Q})/i^*J^m_a(Y)$$

coincides with the image of

$$\mathrm{CH}^m(Y)_{\mathbb{Q}} \to H^{2m}_{\mathcal{D}}(Y,\mathbb{Q}) \to H^{2m}_{\mathcal{D}}(X,\mathbb{Q})/i^*J^m_a(Y).$$

As $H^{0,2m-1}_{\mathrm{var}}(X) \neq 0$ if d is sufficiently large, Theorem 0.0.5 shows that

$$J^m_{\mathrm{alg}}(X) = i^*J^m_{\mathrm{alg}}(Y).$$

Hence the problem is to describe $\overline{\psi}_X(\mathrm{Griff}^m(X)) \subset J^m(X)/i^*J^m_{\mathrm{alg}}(Y)$. The result of Green and Müller–Stach almost achieves this goal: it describes the image of $\overline{\psi}_X$ in the group $J^m(X)/i^*J^m_a(Y)$, which is a quotient of the group $J^m(X)/i^*J^m_{\mathrm{alg}}(Y)$. If the generalized Hodge conjecture for Y holds, then both groups coincide. We denote the Chow group of codimension m cycles on Y whose restriction to X is homologically equivalent to zero by $\mathrm{CH}^m(Y)_0$. Theorem 0.0.6 says that the image of $\mathrm{Griff}^m(X)$ in $J^m(X)/i^*J^m_a(Y)$ comes from elements of $\mathrm{CH}^m(Y)_0$ in one of the following two ways:

1. If $Z \in \mathrm{CH}^m(Y)_0$ and $\mathrm{cl}_Y(Z) \neq 0$, then the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & J^m(Y)_{\mathbb{Q}} & \to & H^{2m}_{\mathcal{D}}(Y,\mathbb{Q}) & \to & \mathrm{Hdg}^m(Y)_{\mathbb{Q}} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & J^m(X)_{\mathbb{Q}} & \to & H^{2m}_{\mathcal{D}}(X,\mathbb{Q}) & \to & \mathrm{Hdg}^m(X)_{\mathbb{Q}} & \to & 0 \end{array}$$

   shows that we obtain an element of $J^m(X)$ by restricting the Deligne cycle class $\mathrm{cl}_{\mathcal{D},Y}(Z)$ to X.

2. If $\mathrm{cl}_Y(Z) = 0$, then we may assume that Z belongs to $\mathrm{Griff}^m(Y)$ (we divide out by the image of $J^m_a(Y) \supset J^m_{\mathrm{alg}}(Y)$), and the restriction of $\psi_Y(Z)$ maps to $J^m(X)/i^* J^m_a(Y)$.

Green and Voisin had previously considered the case $Y = \mathbb{P}^{2m}$ and obtained a sharp degree bound. They showed that the image of the Abel–Jacobi map for a very general hypersurface of degree d is contained in the torsion points of $J^m(X)$ if $d \geq 4 + 2/(m-1)$; see [G3]. Theorem 0.0.6 also holds for complete intersections of sufficiently large multidegree, i.e., complete intersections of multidegree $(d_0, \ldots, d_r)$ such that $\min(d_0, \ldots, d_r)$ is sufficiently large; the proof heavily relies on Nori's connectivity theorem [No].

Remark 0.0.7. It is also possible to define higher order infinitesimal invariants associated to families of algebraic cycles; see [V7]. These invariants belong to the higher Koszul groups $H^i(\mathrm{Gr}^p_F(\bigwedge^\bullet T_0^\vee \otimes H))$ $(i \geq 2)$ and should be related to the conjectural higher cycle class maps ψ_i that we mentioned before. Even though no such relation is known at present, these higher order infinitesimal invariants have been used to study families of zero–cycles on complete intersections; see [V5] and [AS].

In this thesis we study the image of the Abel–Jacobi map for complete intersections of sufficiently large multidegree in a smooth projective variety Y. Our aim is to prove a version of Theorem 0.0.6 with precise degree bounds in case Y is a projective space or, more generally, a Grassmann variety. The basis for our infinitesimal calculations is the description of the variable cohomology of $X = V(d_0, \ldots, d_r) \subset Y$ in terms of a ring R, the Jacobi ring. In Chapter 1 we collect a number of technical results on Jacobi rings. In the case of hypersurfaces, these rings arise by the classical construction of taking residues of differential forms on Y with logarithmic poles along the divisor. Recently the theory of Jacobi rings was extended to the case of complete intersections by a trick that reduces the problem to the codimension one case. The reader may wish to skip Chapter 1 on first reading and refer back to it as needed. In Chapter 2 we generalize the theorem of Green and Voisin to the case of complete intersections in projective space:

Theorem 0.0.8. (cf. Theorem 2.4.1) *Let $X = V(d_0, \ldots, d_r) \subset \mathbb{P}^{2m+r}$ be a smooth complete intersection. Suppose that $m \geq 2$ and $d_0 \geq \ldots \geq d_r$. If X is very general and*

(1) $\sum_{i=0}^{r} d_i + (m-2)d_r \geq 2m + r + 2$

(2) $\sum_{i=1}^{r} d_i + (m-1)d_r \geq 2m + r + 1,$

then the image of $\psi_X : \mathrm{CH}^m_{\mathrm{hom}}(X) \to J^m(X)$ *is contained in the torsion points of* $J^m(X)$.

The exceptional cases of this theorem were known to be counterexamples, except for the case of odd–dimensional complete intersections of four quadrics. We treat this remaining case in Chapter 3:

Theorem 0.0.9. (cf. Theorem 3.3.6) *Let* $X = V(2,2,2,2) \subset \mathbb{P}^{2m+3}$ *(*$m \geq 2$*) be a smooth complete intersection of four quadrics. If* X *is very general, then* $\operatorname{im} \psi_{X,\mathbb{Q}} \neq 0$.

Bardelli [Bar] proved this theorem in the case $m = 2$; the proof for $m > 2$ is based on his methods. Note that as a consequence of this result the degree bounds in Theorem 0.0.8 are sharp.

Chapter 4 is devoted to the generalization of Theorem 0.0.8 to complete intersections in Grassmann varieties.

Theorem 0.0.10. (cf. Theorem 4.3.11) *Let* $X = V(d_0, \ldots, d_r) \subset Y = G(s, \ell+1)$ *(*$d_0 \geq \ldots \geq d_r$*) be a smooth complete intersection of dimension* $2m-1$ *(*$m \geq 2$*). If* X *is very general and*

(1) $\sum_{i=0}^{r} d_i + (m-2)d_r \geq \ell + 2$

(2) $\sum_{i=1}^{r} d_i + (m-1)d_r \geq \ell + 1$

(3) $\sum_{i=\min(1,r)}^{r} d_i \geq \ell - 1,$

then the image of $\mathrm{cl}_{\mathcal{D},X} : \mathrm{CH}^m(X)_{\mathbb{Q}} \to H^{2m}_{\mathcal{D}}(X, \mathbb{Q})$ *coincides with the image of* $i^* \circ \mathrm{cl}_{\mathcal{D},Y} : \mathrm{CH}^m(Y)_{\mathbb{Q}} \to H^{2m}_{\mathcal{D}}(X, \mathbb{Q})$.

Note that condition (3), which is needed for the description of the cohomology of X in terms of the Jacobi ring, is stronger than the conditions (1) and (2) if $m > 2$. In the case of Grassmann varieties of lines in projective space ($s = 2$) we can replace condition (3) by a weaker condition, and we obtain reasonably precise degree bounds; see Proposition 4.4.2. For instance, if $Y = G(2,5)$ is the Grassmann variety of lines in $\mathbb{P}^4$ (the first interesting case, as $G(2,4)$ can be embedded as a smooth quadric in $\mathbb{P}^5$), we obtain the following result:

Theorem 0.0.11. *Let $X = V(d_0, \ldots, d_r)$ ($d_0 \geq \ldots \geq d_r, r \leq 2$) be a smooth complete intersection in $Y = G(2,5)$. Then the conclusion of Theorem 0.0.10 holds for very general X, except possibly if*

(i) $X = V(2)$

(ii) $X = V(d,1,1)$, $d \geq 1$

(iii) $X = V(d,2,1)$, $d \geq 2$.

We consider one of the exceptional cases, the quadratic line complex in the Grassmann variety of lines in $\mathbb{P}^4$, in Chapter 5. This five–dimensional quadratic line complex has been studied by the classical geometers B. Segre and L. Roth. We prove the generalized Hodge conjecture in this case, thus verifying that the quadratic line complex indeed provides a counterexample to the previous theorem:

Theorem 0.0.12. (cf. Theorem 5.3.9) *Let $X = V(2) \subset G(2,5)$ be a smooth quadratic line complex. If X is general, then*

$$\psi_X : \mathrm{CH}^3_{\mathrm{hom}}(X) \to J^3(X)$$

is surjective.

Notation

We work throughout over the field of complex numbers. Cohomology is with coefficients in $\mathbb{C}$, unless stated otherwise.

(i) We say that a general (resp. very general) point of a variety X has property (P) if the set of points $x \in X$ that do not satisfy this property is contained in a Zariski closed proper subset (resp. is contained in a countable union of proper analytic subsets).

(ii) A good compactification of a quasi–projective variety X is a smooth, compact projective variety $\overline{X} \supset X$ such that $\overline{X} \setminus X$ is a divisor with normal crossings.

(iii) Let E be a vector bundle over a smooth projective variety Y. The zero locus of a global section $s \in H^0(Y, E)$ is denoted by $V(s)$. A complete intersection of multidegree $(d_0, \ldots, d_r)$ in a smooth projective variety Y is denoted by $V(d_0, \ldots, d_r)$.

(iv) The Grassmann variety of k–dimensional linear subspaces of a complex vector space V of dimension n is denoted by $G(k, V)$ or $G(k, n)$. The Hilbert scheme of k–planes that are contained in a smooth projective variety X is denoted by $F_k(X)$. By abuse of language, $F_k(X)$ is usually callled the Fano variety of k–planes contained in X.

(v) The words pencil, net and web refer to linear systems of dimension one, two and three respectively. The base locus of a linear system $V \subset |L|$ is denoted by $\mathrm{Bs}(V)$.

(vi) If $V_1, \ldots, V_k$ are linear subspaces of a complex vector space V, their linear span is denoted by $\langle V_1, \ldots, V_k \rangle$.

(vii) We say that a property (P) holds for sufficiently ample line bundles if there exists an ample line bundle L_0 such that (P) holds for all line bundles L such that $L \otimes L_0^{-1}$ is ample.

(viii) Let $f : X \to S$ be a proper surjective morphism of quasi–projective varieties. The locus $\mathcal{C} = \{x \in X : \dim f_*(T_{X,x}) < \dim T_{S,f(x)}\}$ is called the critical locus of f. The image $\Delta = f(\mathcal{C}) \subset S$ is called the discriminant locus of f.

Chapter 1

Jacobi rings

1.1 Introduction

This chapter contains a number of technical results that will be used in Chapters 2 and 4. Let Y be a smooth projective variety of dimension $n+1$, and let $X \subset Y$ be an ample divisor with inclusion map $i : X \to Y$. The cohomology group $H^n(X, \mathbb{Q})$ splits into a 'fixed' part $H^n_{\mathrm{fix}}(X, \mathbb{Q}) = i^* H^n(Y, \mathbb{Q})$ and a 'variable' part $H^n_{\mathrm{var}}(X, \mathbb{Q})$. Using residues of rational differential forms on Y with poles along X, one can describe the graded pieces of the Hodge filtration on $H^n_{\mathrm{var}}(X)$ in terms of a ring R, the Jacobi ring. This construction was classically known for curves in $\mathbb{P}^2$; it was generalized to the case of hypersurfaces in projective space by Griffiths [Gr2], and to the case of sufficiently ample divisors in an arbitrary smooth projective variety Y by Carlson, Green, Griffiths and Harris [CGGH]. The description of $H^n_{\mathrm{var}}(X)$ in terms of the Jacobi ring is useful for infinitesimal computations in Hodge theory.

In Section 1.2 we present the method of [CGGH] in a slightly different way, using jet bundles. The generalization to the case of complete intersections is given in Section 1.3, following [Te2] and [Ko2].

1.2 Jacobi rings for hypersurfaces

Let Y be a smooth projective variety of dimension $n+1$. Let L be a very ample line bundle on Y, and let $X \in |L|$ be a smooth divisor. We denote the inclusion maps by $i : X \to Y$ and $j : Y \setminus X \to Y$. Starting from the long exact sequence of relative homology groups

$$\cdots \to H_{n+1}(Y,\mathbb{Q}) \to H_{n+1}(Y,X;\mathbb{Q}) \to H_n(X,\mathbb{Q}) \to H_n(Y,\mathbb{Q}) \to \cdots$$

we apply Poincaré–Lefschetz duality to obtain the Gysin sequence

$$\cdots \to H^{n+1}(Y,\mathbb{Q}) \xrightarrow{j^*} H^{n+1}(Y\setminus X,\mathbb{Q}) \xrightarrow{\partial} H^n(X,\mathbb{Q}) \xrightarrow{i_*} H^{n+2}(Y,\mathbb{Q}) \to \cdots.$$

The inclusion of complexes

$$\Omega^\bullet_Y(\log X) \to j_*\Omega^\bullet_{Y\setminus X}$$

is a quasi–isomorphism; see [D2, I, Cor. 3.13]. In [PS, (11.4)] it is shown that the Gysin sequence is connected to the long exact sequence of hypercohomology groups associated to the Poincaré residue sequence

$$0 \to \Omega^\bullet_Y \to \Omega^\bullet_Y(\log X) \to i_*\Omega^\bullet_X[-1] \to 0$$

by the commutative diagram

$$\begin{array}{ccccccc}
\mathbb{H}^{n-1}(\Omega^\bullet_X) & \longrightarrow & \mathbb{H}^{n+1}(\Omega^\bullet_Y) & \longrightarrow & \mathbb{H}^{n+1}(\Omega^\bullet_Y(\log X)) & \xrightarrow{\mathrm{Res}} & \mathbb{H}^n(\Omega^\bullet_X) \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
H^{n-1}(X,\mathbb{C}) & \xrightarrow{2\pi i\, i_*} & H^{n+1}(Y,\mathbb{C}) & \longrightarrow & H^{n+1}(Y\setminus X,\mathbb{C}) & \xrightarrow{\frac{1}{2\pi i}\partial} & H^n(X,\mathbb{C}).
\end{array}$$

Let $F^\bullet$ be the 'filtration bête' on $\Omega^\bullet_Y(\log X)$, given by

$$F^p\Omega^q_Y(\log X) = \begin{cases} \Omega^q_Y(\log X) & \text{if } q \geq p \\ 0 & \text{if } q < p. \end{cases}$$

We define an increasing weight filtration $W_\bullet$ on $\Omega^\bullet_Y(\log X)$ by

$$W_k\Omega^p_Y(\log X) = \begin{cases} 0 & \text{if } k < 0 \\ \Omega^p_Y & \text{if } k = 0 \\ \Omega^p_Y(\log X) & \text{if } k \geq 1. \end{cases}$$

Let

$$F^pH^k(Y\setminus X) = \operatorname{im} \mathbb{H}^k(F^p\Omega^\bullet_Y(\log X)) \to \mathbb{H}^k(\Omega^\bullet_Y(\log X))$$

and

$$W_{k+p}H^k(Y\setminus X) = \operatorname{im} \mathbb{H}^k(W_p\Omega^\bullet_Y(\log X)) \to \mathbb{H}^k(\Omega^\bullet_Y(\log X))$$

be the induced filtrations on $\mathbb{H}^k(\Omega^\bullet_Y(\log X)) \cong H^k(Y\setminus X,\mathbb{C})$. The weight filtration $W_\bullet$ is defined over $\mathbb{Q}$ (cf. [St, (2.5)]). Deligne has shown that the

Hodge and weight filtrations $F^\bullet$ and $W_\bullet$ define a mixed Hodge structure (MHS) on $H^k(Y \setminus X)$ such that the exact sequence

$$\cdots \to H^{n+1}(Y) \longrightarrow H^{n+1}(Y \setminus X) \xrightarrow{\text{Res}} H^n(X)(-1) \xrightarrow{i_*} H^{n+2}(Y) \to \cdots$$

becomes an exact sequence of mixed Hodge strucures; see [D3, (3.2.5)] and [D4, (9.2.1.2)]. Note that the graded pieces of the weight filtration are

$$\begin{aligned} \mathrm{Gr}_k^W H^k(Y \setminus X) &\cong \operatorname{im} j^* : H^k(Y) \to H^k(Y \setminus X) \\ \mathrm{Gr}_{k+1}^W H^k(Y \setminus X) &\cong \ker i_* : H^{k-1}(X)(-1) \to H^{k+1}(Y). \end{aligned}$$

There is a perfect pairing between the groups $H^k(Y, X; \mathbb{Q}) \cong H_c^k(Y \setminus X, \mathbb{Q})$ and $H^{2n-k+2}(Y \setminus X, \mathbb{Q})$ (Poincaré–Lefschetz duality). Therefore we can also work with the exact sequence

$$\cdots \to H^n(Y, X; \mathbb{Q}) \to H^n(Y, \mathbb{Q}) \to H^n(X, \mathbb{Q}) \to H^{n+1}(Y, X; \mathbb{Q}) \to \cdots$$

that is dual to the Gysin sequence. In order to describe the MHS on the relative cohomology $H^k(Y, X)$, we recall the definition of the mapping cone of a morphism of complexes.

Definition 1.2.1. *Let $f : A^\bullet \to B^\bullet$ be a morphism of complexes. The complex $C^\bullet(f) = A^\bullet \oplus B^\bullet[-1]$ is called the mapping cone of f. Its differential is given by $d(\alpha, \beta) = (d_A(\alpha), f(\alpha) - d_B(\beta))$.*

The complex $C^\bullet(f)$ fits into an exact sequence

$$0 \to B^\bullet[-1] \to C^\bullet(f) \to A^\bullet \to 0$$

and the connecting homomorphism $H^k(A^\bullet) \to H^{k+1}(B^\bullet[-1]) \cong H^k(B^\bullet)$ coincides with the map $H^k(f)$ induced by f.

Remark 1.2.2. Some authors define $C^\bullet(f) = A^\bullet[1] \oplus B^\bullet$. We have adopted a different convention here, since it leads to a more natural indexation. With this convention, $C^\bullet(f)$ is isomorphic to the total complex associated with the double complex $A^\bullet \to B^\bullet$, where $A^\bullet$ is put in degree zero.

Let $f : \Omega_Y^\bullet \to i_*\Omega_X^\bullet$ be the restriction map. The complex $C^\bullet(f)$ is quasi–isomorphic to

$$\Omega_{Y,X}^\bullet = \ker f : \Omega_Y^\bullet \to i_*\Omega_X^\bullet.$$

The five lemma shows that $\mathbb{H}^k(C^\bullet(f)) \cong H^k(Y, X; \mathbb{C})$. Set

$$\begin{array}{rcl} W_{-1}C^\bullet(f) & = & i_*\Omega_X^\bullet[-1] \\ W_0 C^\bullet(f) & = & C^\bullet(f), \end{array}$$

and let $F^\bullet$ be the filtration bête on $C^\bullet(f)$. The induced filtrations $W_\bullet$ and $F^\bullet$ on the cohomology define a MHS on $H^k(Y, X)$ (cf. [DD, Lemme 2.2]). Note that

$$\begin{array}{rcl} \mathrm{Gr}_{k-1}^W H^k(Y, X) & \cong & \mathrm{coker}\, i^* : H^{k-1}(Y) \to H^{k-1}(X) \\ \mathrm{Gr}_k^W H^k(Y, X) & \cong & \ker i^* : H^k(Y) \to H^k(X). \end{array}$$

Remark 1.2.3.

(i) The duality isomorphism

$$D : H^{2n-k+2}(Y, X; \mathbb{Q}) \to \mathrm{Hom}(H^k(Y \setminus X, \mathbb{Q}), \mathbb{Q})(-n-1)$$

is an isomorphism of MHS. The isomorphism between

$$\mathrm{Gr}_F^p H^{2n-k+2}(Y, X) \cong H^{2n-k-p+2}(Y, \Omega_{Y,X}^p)$$

and

$$\begin{array}{rcl} \mathrm{Gr}_F^p H^k(Y \setminus X)^\vee(-n-1) & \cong & (\mathrm{Gr}_F^{n+1-p} H^k(Y \setminus X))^\vee \\ & \cong & H^{k+p-n-1}(Y, \Omega_Y^{n-p+1}(\log X))^\vee \end{array}$$

can be interpreted in terms of Serre duality: we have a nondegenerate pairing

$$\Omega_Y^p(\log X) \times \Omega_{Y,X}^{n-p+1} \to K_Y$$

given by wedge product. Let

$$T_Y(-\log X) = \{\theta \in \mathrm{Der}(\mathcal{O}_Y) : \theta.\mathcal{I}_X \subseteq \mathcal{I}_X\}$$

be the sheaf of vector fields on Y that are tangent to X. The sheaf $T_Y(-\log X)$ is dual to $\Omega_Y^1(\log X)$, hence locally free of rank $n+1$. If we combine the isomorphism

$$K_Y \otimes \bigwedge^p T_Y(-\log X) \cong \Omega_{Y,X}^{n-p+1}$$

with Serre duality, we obtain a perfect pairing

$$H^q(Y, \Omega_Y^p(\log X)) \times H^{n-q+1}(Y, \Omega_{Y,X}^{n-p+1}) \to H^{n+1}(Y, \Omega_Y^{n+1}) \cong \mathbb{C}.$$

(ii) Note that $\Omega_Y^{n+1}(\log X) \cong K_Y \otimes L$; hence $\Omega_{Y,X}^{n-p+1} \cong \Omega_Y^{n-p+1}(\log X) \otimes L^{-1}$.

Cup product with $c_1(L)$ defines the Lefschetz operator

$$u_Y : H^k(Y) \to H^{k+2}(Y).$$

For $k \leq n+1$ we define the primitive part of the cohomology of Y by

$$H^k_{\mathrm{pr}}(Y) = \ker u_Y^{n-k+2} : H^k(Y) \to H^{2n-k+4}(Y).$$

The primitive cohomology groups $H^k_{\mathrm{pr}}(X)$ are defined analogously, by means of the Lefschetz operator u_X.

Definition 1.2.4. *The variable cohomology of X is*

$$H^n_{\mathrm{var}}(X) = \ker i_* : H^n(X) \to H^{n+2}(Y).$$

Let $Q_X(\,,)$ (resp. $Q_Y(\,,)$) be the nondegenerate bilinear form on the vector space $H^n(X,\mathbb{Q})$ (resp. $H^{n+2}(Y,\mathbb{Q})$) given by cup product.

Lemma 1.2.5. *There is an isomorphism of $\mathbb{Q}$–Hodge structures*

$$H^n_{\mathrm{pr}}(X,\mathbb{Q}) \cong i^* H^n_{\mathrm{pr}}(Y,\mathbb{Q}) \bigoplus H^n_{\mathrm{var}}(X,\mathbb{Q}).$$

Proof: If $\alpha \in H^n(X,\mathbb{Q})$ and $\beta \in H^n(Y,\mathbb{Q})$, then $Q_X(\alpha, i^*\beta) = Q_Y(i_*\alpha, \beta)$. Therefore $H^n_{\mathrm{var}}(X,\mathbb{Q})$ is the orthogonal complement of $i^*H^n(Y,\mathbb{Q})$ with respect to $Q_X(\,,)$. The commutative diagram

$$\begin{array}{ccccc}
H^{n-2}(Y,\mathbb{Q}) & \xrightarrow{u_Y} & H^n(Y,\mathbb{Q}) & \xrightarrow{u_Y} & H^{n+2}(Y,\mathbb{Q}) \\
\downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\
H^{n-2}(X,\mathbb{Q}) & \xrightarrow{u_X} & H^n(X,\mathbb{Q}) & \xrightarrow{u_X} & H^{n+2}(X,\mathbb{Q})
\end{array}$$

shows that $u_X H^{n-2}(X,\mathbb{Q}) \subset i^*H^n(Y,\mathbb{Q})$, since the vertical arrow on the left hand side is an isomorphism by the Lefschetz hyperplane theorem. Thus $H^n_{\mathrm{var}}(X,\mathbb{Q}) \subset H^n_{\mathrm{pr}}(X,\mathbb{Q})$. Since the map $i_* : H^{n+2}(X,\mathbb{Q}) \to H^{n+4}(Y,\mathbb{Q})$ is injective and $i_* u_X i^* = u_Y^2$, an element $\beta \in H^n(Y,\mathbb{Q})$ belongs to $H^n_{\mathrm{pr}}(Y,\mathbb{Q}) = \ker u_Y^2$ if and only if $i^*\beta$ belongs to $H^n_{\mathrm{pr}}(X,\mathbb{Q}) = \ker u_X$. Thus $i^*H^n(Y,\mathbb{Q}) \cap H^n_{\mathrm{pr}}(X,\mathbb{Q}) = i^*H^n_{\mathrm{pr}}(Y,\mathbb{Q})$. As $H^n_{\mathrm{pr}}(X,\mathbb{Q})$ carries a polarized Hodge structure, the orthogonal complement $H^n_{\mathrm{var}}(X,\mathbb{Q})$ of $i^*H^n_{\mathrm{pr}}(Y,\mathbb{Q})$ in $H^n_{\mathrm{pr}}(X,\mathbb{Q})$ is a sub Hodge structure; thus we obtain the desired splitting. $\square$

The exact sequence

$$0 \to \Omega_Y^{n-p+1} \to \Omega_Y^{n-p+1}(\log X) \to i_*\Omega_X^{n-p} \to 0$$

induces a short exact sequence

$$0 \to H^p_{\mathrm{pr}}(Y, \Omega_Y^{n-p+1}) \to H^p(Y, \Omega_Y^{n-p+1}(\log X)) \to H^p_{\mathrm{var}}(X, \Omega_X^{n-p}) \to 0.$$

In addition to the Hodge filtration $F^\bullet$ there is another natural filtration on $H^{n+1}(Y \setminus X)$, the order of pole filtration $P^\bullet$. We define $P^p j_*\Omega^\bullet_{Y\setminus X} \subset j_*\Omega^\bullet_{Y\setminus X}$ as the subcomplex

$$0 \to \cdots \to 0 \to \Omega_Y^p(X) \to \Omega_Y^{p+1}(2X) \to \cdots$$

that starts in degree p. Deligne has shown that the inclusion of complexes

$$(\Omega_Y^\bullet(\log X), F^\bullet) \to (j_*\Omega^\bullet_{Y\setminus X}, P^\bullet)$$

is a filtered quasi–isomorphism; see [D2, I, Prop. 3.13]. Given suitable vanishing theorems of the form

$$H^i(Y, \Omega_Y^j(kX)) = 0,$$

we obtain an isomorphism

$$H^p(\Omega_Y^{n+1-p}(\log X)) \cong \frac{H^0(K_Y \otimes L^{p+1})}{H^0(K_Y \otimes L^p) + dH^0(\Omega_Y^n \otimes L^p)} \tag{1.1}$$

that describes the graded pieces of the Hodge filtration on $H^{n+1}(Y \setminus X)$ in terms of rational $(n+1)$–forms on Y with poles along X (see [CGGH, §3] or [PS, §11]).

In the case of degree d hypersurfaces $X = V(f) \subset \mathbb{P}^{n+1}$ this approach was worked out by Griffiths [Gr2]; it leads to an isomorphism

$$H^{n-p,p}_{\mathrm{var}}(X) \cong R_{(p+1)d-n-2}$$

where R, the Jacobi ring, is the quotient of $\mathbb{C}[x_0, \dots, x_{n+1}]$ by the Jacobi ideal $J = (\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_{n+1}})$.

Embed Y as a subvariety in $\mathbb{P}H^0(Y, L)^\vee$ by the linear system $|L|$, and let

$$\tilde{S} = \bigoplus\nolimits_{p\geq 0} H^0(Y, L^p)$$

be its homogeneous coordinate ring. The $\tilde{S}$–module

$$M = \bigoplus\nolimits_{p\geq 0} H^0(Y, K_Y \otimes L^{p+1})$$

is called the *Arbarello–Sernesi module*; it contains the *Jacobi module*

$$J = \bigoplus_{p\geq 0} sH^0(Y, K_Y \otimes L^p) + dH^0(Y, \Omega^n_Y \otimes L^p)$$

as a submodule. The isomorphism (1.1) leads to description of the variable cohomology of X in terms of the quotient module M/J. We shall take a slightly different approach, based on an elegant description of J in terms of jet bundles. This is due to Green and Griffiths (see [G4, Lecture 4]; cf. also [G1] and [SSU]).

Let $\Delta(2)$ be the subscheme of $Y \times Y$ defined by $\mathcal{I}_\Delta^2$, where $\mathcal{I}_\Delta$ is the ideal sheaf of the diagonal $\Delta \subset Y \times Y$. Let p_1 and p_2 be the projections of $Y \times Y$ onto the first and second factor. The sheaf

$$P^1(L) = (p_1)_*(p_2^* L \otimes \mathcal{O}_{\Delta(2)})$$

fits into an exact sequence of $\mathcal{O}_Y$–modules

$$0 \to \Omega^1_Y \otimes L \to P^1(L) \to L \to 0 \tag{1.2}$$

and is thus locally free of rank $n+2$ (cf. [EGA] or [Kl, IV A]). The extension class of (1.2) is $2\pi i.c_1(L) \in H^1(Y, \Omega^1_Y)$; see [At, Prop. 12]. There is a natural mapping

$$j^1 : L \to P^1(L)$$

that sends a section s to its 1–jet $j^1(s)$. Let

$$\Sigma_L = \mathrm{Hom}(P^1(L), L) = P^1(L)^\vee \otimes L$$

be the bundle of first order differential operators on sections of L. Dualizing (1.2) and twisting by L, we obtain an exact sequence

$$0 \to \mathcal{O}_Y \to \Sigma_L \to T_Y \to 0 \tag{1.3}$$

with extension class $-2\pi i.c_1(L)$.

Remark 1.2.6. If $Y = \mathbb{P}^{n+1}$, the sequence (1.3) coincides with the Euler sequence. This follows from [Kl, (IV,17)].

Lemma 1.2.7. *If $X = V(s) \subset Y$ is a smooth divisor, we have an exact sequence*

$$0 \to T_Y(-\log X) \to \Sigma_L \to L \to 0, \tag{1.4}$$

where the map $T_Y(-\log X) \to \Sigma_L$ is the natural inclusion of $T_Y(-\log X)$ in the sheaf of first order differential operators on sections of L and the map $\Sigma_L \to L$ is given by contraction with $j^1(s)$.

Proof: (cf. [SSU, I, (6.2.4)] or [Ko2, Lemma 2.3]) Since $X \subset Y$ is a smooth divisor, we can choose local coordinates $(z_1, \ldots, z_{n+1})$ on Y such that $L|_U \cong U \times \mathbb{C}$ and $s(z_1, \ldots, z_{n+1}) = z_1$. A local frame for $T_Y(-\log X)$ is given by

$$z_1 \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_{n+1}}.$$

If

$$D = a(z) + \sum_{i=1}^{n+1} b_i(z) \frac{\partial}{\partial z_i} \in \Gamma(U, \Sigma_L)$$

is a first order differential operator then

$$\langle D, j^1(s) \rangle = D(s) = a(z)s + \sum_{i=1}^{n+1} b_i(z) \frac{\partial s}{\partial z_i} = a(z)s + b_1(z).$$

Hence the symbol map $\Sigma_L \to T_Y$ induces an isomorphism between the kernel of $j^1(s) : \Sigma_L \to L$ and $T_Y(-\log X)$. $\square$

We can now define the generalization of the Jacobi module J.

Definition 1.2.8. *Let $X = V(s) \subset Y$ be a smooth divisor, and let $\mathcal{F}$ be a coherent sheaf on Y.*
(i) $\tilde{J}_{Y,s}(\mathcal{F})$ is the image of the map

$$H^0(Y, \mathcal{F} \otimes \Sigma_L \otimes L^{-1}) \to H^0(Y, \mathcal{F})$$

that is induced by the exact sequence (1.4).
(ii)

$$\tilde{R}_{Y,s}(\mathcal{F}) = H^0(Y, \mathcal{F}) / \tilde{J}_{Y,s}(\mathcal{F}).$$

The natural candidate for the Jacobi ring is the ring

$$\tilde{R} = \bigoplus\nolimits_{p,q \geq 0} \tilde{R}_{Y,s}(K_Y \otimes L^{p+1}).$$

In general this is *not* the right choice; we need a slightly modified version. Define

$$R_{Y,s}(K_Y \otimes L^{p+1}) = \begin{cases} \tilde{R}_{Y,s}(K_Y \otimes L^{p+1}) & \text{if } p \geq 1 \\ H^0(Y, K_Y \otimes L) & \text{if } p = 0. \end{cases}$$

Similarly we define

$$J_{Y,s}(K_Y \otimes L^{p+1}) = \begin{cases} \tilde{J}_{Y,s}(K_Y \otimes L^{p+1}) & \text{if } p \geq 1 \\ 0 & \text{if } p = 0. \end{cases}$$

Definition 1.2.9.

(i)
$$S = \bigoplus_{p,q\geq 0} H^0(Y, K_Y^{\otimes q} \otimes L^{p+1}).$$

(ii) The ring
$$R = \bigoplus_{p,q\geq 0} R_{Y,s}(K_Y^{\otimes q} \otimes L^{p+1})$$
is the Jacobi ring of X.

(iii) The ideal
$$J = \bigoplus_{p,q\geq 0} J_{Y,s}(K_Y^{\otimes q} \otimes L^{p+1})$$
is the Jacobi ideal of X.

Remark 1.2.10. If $H^0(Y, K_Y) = H^0(Y, \Omega_Y^n) = 0$, then $\tilde{J}_{Y,s}(K_Y \otimes L) = 0$ and the rings R and $\tilde{R}$ coincide. This happens for instance when Y is a Grassmann variety or a quadric.

Lemma 1.2.11. *If $p > 0$ and*

(i) $H^i(Y, \Omega_Y^{n+1-i} \otimes L^{p+1-i}) = 0$ *for all* $1 \leq i \leq p$

(ii) $H^i(Y, \Omega_Y^{n+1-i} \otimes L^{p-i}) = H^i(Y, \Omega_Y^{n-i} \otimes L^{p-i}) = 0$ *for all* $1 \leq i \leq p-1$,

then
$$H^p(Y, \Omega_Y^{n-p+1}(\log X)) \cong R_{Y,s}(K_Y \otimes L^{p+1}).$$
If $p = 0$, then
$$H^0(Y, \Omega_Y^{n+1}(\log X)) \cong H^0(Y, K_Y \otimes L).$$

Proof: Dualize (1.4) and take exterior powers to obtain a resolution

$$0 \to \Omega_Y^{n-p+1}(\log X) \to \bigwedge^{n-p+2} \Sigma_L^\vee \otimes L \to \cdots \to \bigwedge^{n+2} \Sigma_L^\vee \otimes L^{p+1} \to 0.$$

Set
$$C^k = \begin{cases} \Omega_Y^{n-p+1}(\log X) & \text{if } k = 0 \\ \wedge^{n-p+k+1} \Sigma_L^\vee \otimes L^k & \text{if } 1 \leq k \leq p+1. \end{cases}$$
The spectral sequence

$$E_1^{p,q} = H^q(Y, C^p) \Rightarrow \mathbb{H}^{p+q}(C^\bullet)$$

associated to this complex converges to zero, since the complex is exact. Chasing through this spectral sequence, we find that

$$\begin{aligned} H^p(\Omega_Y^{n-p+1}(\log X)) &\cong \text{coker } H^0(\bigwedge^{n+1} \Sigma_L^\vee \otimes L^p) \to H^0(\bigwedge^{n+2} \Sigma_L^\vee \otimes L^{p+1}) \\ &= R_{Y,s}(K_Y \otimes L^{p+1}) \end{aligned}$$

if $p > 0$ and

$$\begin{aligned} H^1(Y, \textstyle\bigwedge^{n+1}\Sigma_L^\vee \otimes L^p) = \ldots = H^p(Y, \textstyle\bigwedge^{n-p+2}\Sigma_L^\vee \otimes L) = 0 \\ H^1(Y, \textstyle\bigwedge^{n}\Sigma_L^\vee \otimes L^{p-1}) = \ldots = H^{p-1}(Y, \textstyle\bigwedge^{n-p+2}\Sigma_L^\vee \otimes L) = 0. \end{aligned}$$

The result then follows from the exact sequence

$$0 \to \Omega_Y^k \to \textstyle\bigwedge^k \Sigma_L^\vee \to \Omega_Y^{k-1} \to 0$$

that is obtained from (1.3) by dualizing and taking exterior powers. Note that

$$H^i(Y, \Omega_Y^{n+2-i} \otimes L^{p+1-i}) = 0 \text{ if } 1 \leq i \leq p$$

by the Kodaira–Nakano vanishing theorem, since L is ample. For the case $p = 0$, note that $\Omega_Y^{n+1}(\log X) \cong K_Y \otimes L$. $\square$

For $k \geq n+1$ we define

$$\begin{aligned} H^k(Y)_0 &= \ker u_Y : H^k(Y) \to H^{k+2}(Y) \\ &\cong \ker i^* : H^k(Y) \to H^k(X). \end{aligned}$$

The last equality follows from the commutative diagram

$$\begin{tikzcd} H^k(Y) \arrow[r, "u_Y"] \arrow[d, "i^*"] & H^{k+2}(Y) \\ H^k(X) \arrow[ur, "i_*"'] & \end{tikzcd}$$

since the map i_* is an isomorphism if $k \geq n+1$. Note that $H^{n+1}(Y)_0 = H^{n+1}_{\mathrm{pr}}(Y)$.

Remark 1.2.12. If $Y = \mathbb{P}^{n+1}$ and $L = \mathcal{O}_{\mathbb{P}}(d)$, there is a duality between $R_{Y,s}(K_Y \otimes L^{p+1})$ and $R_{Y,s}(K_Y \otimes L^{n-p+1})$. In general this duality fails, due to the presence of primitive cohomology on Y: it can be shown that if L is sufficiently ample, there is a map $R_{Y,s}(K_Y \otimes L^{p+1}) \to R_{Y,s}(K_Y \otimes L^{n-p+1})^\vee$ with kernel $H^{n-p+1,p}_{\mathrm{pr}}(Y)$ and cokernel $H^{n-p,p+1}_{\mathrm{pr}}(Y)$.

Since $H^1(Y, K_Y \otimes L^p) = 0$ for all $p \geq 1$ by the Kodaira vanishing theorem, the tensor product of (1.3) with $K_Y \otimes L^p$ gives rise to a short exact sequence

$$0 \to H^0(Y, K_Y \otimes L^p) \to H^0(Y, \Sigma_L \otimes K_Y \otimes L^p) \to H^0(Y, \Omega_Y^n \otimes L^p) \to 0.$$

This sequence fits into a commutative diagram

$$\begin{tikzcd}
0 & & \\
H^0(Y,\Omega^n_Y\otimes L^p) \arrow[u] \arrow[dr, "d"] & & \\
H^0(Y,\Sigma_L\otimes K_Y\otimes L^p) \arrow[u] \arrow[r, "j^1(s)"] & H^0(Y,K_Y\otimes L^{p+1}). \\
H^0(Y,K_Y\otimes L^p) \arrow[u] \arrow[ur, "s"'] & & \\
0 \arrow[u] & &
\end{tikzcd}$$

Hence we obtain an isomorphism

$$R_{Y,s}(K_Y\otimes L^{p+1})\cong \frac{H^0(Y,K_Y\otimes L^{p+1})}{H^0(Y,K_Y\otimes L^p)+dH^0(Y,\Omega^n_Y\otimes L^p)}.$$

If the conditions of Lemma 1.2.11 are satisfied, we recover the isomorphism (1.1).

Remark 1.2.13. If $X=V(f)\subset\mathbb{P}^{n+1}$, $\deg f=d$, the ring R is graded in a natural way by $\mathrm{Pic}(\mathbb{P}^{n+1})=\mathbb{Z}$ and becomes a zero–dimensional Gorenstein ring of top degree $(n+2)(d-2)$. Since the sequence (1.3) coincides with the Euler sequence (see Remark 1.2.6), we recover the usual definition of the Jacobi ideal: the graded piece J_k is the image of the map

$$\left(\tfrac{\partial f}{\partial x_0},\dots,\tfrac{\partial f}{\partial x_{n+1}}\right):\bigoplus^{n+1}H^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(k+1-d))\to H^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(k)).$$

See also [G4, Lecture 4].

1.3. Jacobi rings for complete intersections

Having treated the case of divisors, we turn to complete intersections in Y. Suppose that $\dim Y=n+r+1$ $(r>0)$. Let

$$E=\bigoplus\nolimits_{i=0}^{r}L_i$$

be a direct sum of very ample line bundles on Y, and let $X = V(s) \subset Y$ be a smooth n–dimensional complete intersection of divisors $D_i \in |L_i|$ defined by a section $s \in H^0(Y, E)$. Let $i : X \to Y$ be the inclusion map, and define $H^n_{\mathrm{var}}(X) = \ker i_* : H^n(X) \to H^{n+2r+2}(Y)$. In this case, the Gysin sequence

$$\to H^{n+2r+1}(Y \setminus X) \to H^n(X) \to H^{n+2r+2}(Y) \to H^{n+2r+2}(Y \setminus X) \to$$

does not lead to a direct description of $H^n_{\mathrm{var}}(X)$, as we have to choose a good compactification of $Y \setminus X$ to describe the MHS on $H^{n+2r+1}(Y \setminus X)$.

If we assume that

(∗) $D = D_0 \cup \ldots \cup D_r$ is a strict normal crossing divisor,

the MHS on the cohomology of $Y \setminus D$ is well understood. One can then study $H^n_{\mathrm{var}}(X)$ using the residue map

$$H^{n+r+1}(Y \setminus D) \to H^n_{\mathrm{var}}(X).$$

This approach has been worked out by Libgober [Li]; it leads to a description of $H^n_{\mathrm{var}}(X)$ in terms of residues of meromorphic differential forms on Y, but the assumption (∗) is too restrictive.

There is a method that reduces the description of $H^n_{\mathrm{var}}(X)$ to the case of divisors. For complete intersections in projective space this is essentially due to Terasoma [Te2]. See the introduction of Chapter 2 for a more detailed discussion of the origins of this method.

The idea is to reduce the problem to the case of a divisor in the projective bundle $P = \mathbb{P}(E^\vee)$ of hyperplanes in the fibers of E. Let $\xi_E = \mathcal{O}_P(1)$ be the tautological line bundle on P, and let $\pi : P \to Y$ be the projection map. As in the case of projective space we have

$$R^i\pi_*\xi_E^k = \begin{cases} S^kE & \text{if } \ i = 0, \ k \geq 0 \\ \det E^\vee \otimes S^{-k-r-1}E^\vee & \text{if } \ i = r, \ k \leq -r-1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\sigma \in H^0(P, \xi_E)$ be the section that corresponds to $s \in H^0(Y, E)$ under the canonical isomorphism $H^0(P, \xi_E) = H^0(Y, E)$. The section σ defines a divisor

$$\mathcal{X} = V(\sigma) \subset P$$

whose Hodge–theoretic properties are strongly related to that of $X \subset Y$.

Lemma 1.3.1. *ξ_E is very ample if and only if the line bundles $L_0, \ldots, L_r$ are very ample.*

Proof: (cf. [BeS, (3.2.3)]) We identify Y with the zero section in the line bundle $L_i^\vee$ ($i = 0, \ldots, r$). By definition, ξ_E is very ample if and only if

$$(f_0, \ldots, f_r) : E^\vee \setminus Y \to H^0(Y, E)^\vee \setminus \{0\} \text{ is an embedding.}$$

Hence ξ_E is very ample if and only if the map $f_i : L_i^\vee \setminus Y \to H^0(Y, L_i)^\vee \setminus \{0\}$ is an embedding for all $i = 0, \ldots, r$, and the assertion follows. $\square$

Lemma 1.3.2.

X is smooth of dimension n $\iff$ $\mathcal{X}$ is smooth of dimension $n + 2r$.

Proof: Choose a local frame $\{e_0, \ldots, e_r\}$ of E over an open subset $U \subset Y$. We can express the section $s_U = s|_U$ in terms of this local frame as $s_U = \sum_{i=0}^r s_i e_i$. The corresponding section $\sigma_U : U \times \mathbb{P}^r \to \mathbb{C}$ is given by the same expression, where we consider the $e_i(y)$ as coordinate functions on the fiber $E_y^\vee$. Choose local coordinates $(y_1, \ldots, y_{n+r+1})$ on U and homogeneous coordinates $(z_0 : \ldots : z_r)$ on $\mathbb{P}^r$. We can write

$$\sigma_U(y_1, \ldots, y_{n+r+1}, z_0 : \ldots : z_r) = \sum_{i=0}^r s_i(y) z_i.$$

The partial derivatives

$$\frac{\partial \sigma_U(y,z)}{\partial y_k} = \sum_{i=0}^r \frac{\partial s_i(y)}{\partial y_k} z_i$$

and

$$\frac{\partial \sigma_U(y,z)}{\partial z_k} = s_k(y)$$

simultaneously vanish at (y, z) if and only if

1. $s_0(y) = \ldots = s_r(y) = 0$

2. $\left(\dfrac{\partial s_i(y)}{\partial y_j} \right)$ has rank at most r.

Therefore $(y, z) \in \mathcal{X}$ is singular if and only if $y \in X$ is singular. $\square$

Lemma 1.3.3.

$$H^k_{\mathrm{pr}}(P,\mathbb{Q}) \cong \begin{cases} H^k(Y) & \text{if } 0 \le k \le 2r+1 \\ H^k(Y)/u_Y^{r+1}H^{k-2r-2}(Y) & \text{if } 2r+2 \le k \le n+2r+1. \end{cases}$$

Proof: Let u_ξ be the Lefschetz operator on P, given by cup product with $c_1(\xi_E)$. Set $x = c_1(E)$. We can determine the action of u_ξ on $H^*(P,\mathbb{Q})$ using the isomorphism of rings

$$H^*(P,\mathbb{Q}) \cong \frac{H^*(Y,\mathbb{Q})[x]}{(x^{r+1} + c_1(E)x^r + \ldots + c_{r+1}(E))}.$$

If $k \le 2r+1$, then u_ξ acts as

$$u_\xi(\alpha_{k-2} + \alpha_{k-4}x + \ldots + \alpha_0 x^{[\frac{k-2}{2}]}) = \alpha_{k-2}x + \ldots + \alpha_0 x^{[\frac{k}{2}]}.$$

Thus $H^k_{\mathrm{pr}}(P) \cong H^k(Y) \otimes H^0(\mathbb{P}^r) \cong H^k(Y)$ if $k \le 2r+1$. If $k \ge 2r+2$ then

$$u_\xi(\alpha_{k-2} + \alpha_{k-4}x + \ldots + \alpha_{k-2r-2}x^r) = -c_{r+1}(E) \cup \alpha_{k-2r-2} + \\ +(\alpha_{k-2} - c_r(E) \cup \alpha_{k-2r-2})x + \ldots + (\alpha_{k-2r-4} - c_1(E) \cup \alpha_{k-2r-2})x^r.$$

Set $h = c_1(\mathcal{O}_Y(1))$. Since the Chern polynomial of E is

$$c(E) = (1 + d_0 ht)(1 + d_1 ht)\ldots(1 + d_r ht),$$

the ith Chern class of E is $c_i(E) = s_i(d_0,\ldots,d_r)h^i$, where s_i is the ith symmetric polynomial in $d_0,\ldots,d_r$. Hence we obtain an isomorphism

$$H^k_{\mathrm{pr}}(P,\mathbb{Q}) \cong H^k(Y,\mathbb{Q})/u_Y^{r+1}H^{k-2r-2}(Y,\mathbb{Q}).$$

□

Lemma 1.3.4. *There are isomorphisms of Hodge structures*

(1) $H^{n+2r+1}_{\mathrm{pr}}(P) \cong H^{n+1}_{\mathrm{pr}}(Y)(-r)$.

(2) $H^{n+2r}_{\mathrm{var}}(\mathcal{X}) \cong H^n_{\mathrm{var}}(X)(-r)$.

Proof: Assertion (1) follows from Lemma 1.3.3:

$$\begin{aligned} H^{n+2r+1}_{\mathrm{pr}}(P) &\cong H^{n+2r+1}(Y)/u_Y^{r+1}H^{n-1}(Y) \\ &\cong u_Y^r H^{n+1}_{\mathrm{pr}}(Y). \end{aligned}$$

Since

$$\pi : P \setminus \mathcal{X} \to Y \setminus X$$

is a $\mathbb{C}^r$–bundle, the map π_* induces an isomorphism

$$H_c^{k+2r}(P \setminus \mathcal{X}, \mathbb{Q}) \cong H_c^k(Y \setminus X, \mathbb{Q})$$

for all $k \geq 0$. The isomorphism (2) of variable cohomology groups follows from the commutative diagram

$$\begin{array}{ccccc} H^{n+2r}_{\mathrm{var}}(\mathcal{X},\mathbb{Q}) & \to & H_c^{n+2r+1}(P\setminus\mathcal{X},\mathbb{Q}) & \to & H^{n+2r+1}_{\mathrm{pr}}(P,\mathbb{Q}) \\ \downarrow & & \| & & \| \\ H^n_{\mathrm{var}}(X,\mathbb{Q})(-r) & \to & H_c^{n+1}(Y\setminus X,\mathbb{Q})(-r) & \to & H^{n+1}_{\mathrm{pr}}(Y,\mathbb{Q})(-r) \end{array}$$

and is obtained as the composition of the maps

$$H^{n+2r}(\mathcal{X},\mathbb{Q}) \to H^{n+2r}(\tilde{X},\mathbb{Q}) \xrightarrow{\pi_*} H^n(X,\mathbb{Q}),$$

where $\tilde{X} = \pi^{-1}(X) = X \times \mathbb{P}^r$. □

We shall translate the conditions of Lemma 1.2.11 for the pair $(P, \mathcal{X})$ to conditions on Y; if these are satisfied, the graded pieces of the Hodge filtration on $H^n_{\mathrm{var}}(X)$ are described by the graded pieces of the Jacobi ring $R_{P,\sigma}$ of $\mathcal{X} \subset P$. These results can be found in [Ko2].

On the projective bundle P there is an exact sequence of tangent bundles

$$0 \to T_v \to T_P \to \pi^* T_Y \to 0. \tag{1.5}$$

The relative tangent bundle T_v fits in the relative Euler sequence

$$0 \to \mathcal{O}_P \to \pi^* E^\vee \otimes \xi_E \to T_v \to 0. \tag{1.6}$$

Lemma 1.3.5. *$H^i(P, \Omega^j_P \otimes \xi^k_E) = 0$ if*

$$H^{i+t}(P, \pi^*(\Omega^e_Y \otimes \textstyle\bigwedge^{f+t+1} E) \otimes \xi_E^{k-f-t-1}) = 0$$

for all nonnegative integers e, f and t such that $e + f = j$ and $f + t \leq r$.

Proof: The exact sequence (1.5) induces a filtration on Ω^j_P with graded pieces $\pi^*\Omega^e_Y \otimes \Omega^f_v$ $(e + f = j)$. Hence $H^i(P, \Omega^j_P \otimes \xi^k_E) = 0$ if

$$H^i(P, \pi^*\Omega^e_Y \otimes \Omega^f_v \otimes \xi^k_E) = 0$$

for all nonnegative integers e and f such that $e + f = j$. By dualizing and taking exterior powers, we obtain from the relative Euler sequence (1.6) a resolution

$$0 \to \pi^* \textstyle\bigwedge^{r+1} E^\vee \otimes \xi_E^{-r-1} \to \cdots \to \pi^* \textstyle\bigwedge^{f+1} E \otimes \xi_E^{-f-1} \to \Omega^f_v \to 0.$$

The desired vanishing statement follows by chasing the associated spectral sequence of hypercohomology (or simply by breaking up the resolution into short exact sequences). □

Remark 1.3.6. If $f = 0$, we chase through the whole Koszul resolution of $\mathcal{O}_Y$ in the condition of Lemma 1.3.5. We can simply replace this condition by $H^i(Y, \Omega_Y^j \otimes S^k E) = 0$.

Lemma 1.3.7. *If*

(1) $H^{p-\nu}(Y, \Omega_Y^{n+r+1-p+\nu} \otimes \det E \otimes S^\nu E) = 0$ *for all* $0 \leq \nu \leq p-1$

(2) $H^{p-\nu-1}(Y, \Omega_Y^{n+r+2-p+\nu} \otimes \det E \otimes S^\nu E) = 0$ *for all* $0 \leq \nu \leq p-2$

(3) $H^{p-\nu-1}(Y, \Omega_Y^{n+r+1-p+\nu} \otimes \det E \otimes S^\nu E) = 0$ *for all* $0 \leq \nu \leq p-2$

(4) $H^{p-\nu}(Y, \Omega_Y^{n+r+1-p+\nu} \otimes \bigwedge^r E \otimes S^\nu E) = 0$ *for all* $\max(0, 1-r) \leq \nu \leq p-1$

then

$$H^{p+r}(P, \Omega_P^{n-p+r+1}(\log \mathcal{X})) \cong R_{P,\sigma}(K_P \otimes \xi_E^{p+r+1}).$$

Proof: Lemma 1.2.11 shows that

$$H^{p+r}(P, \Omega_P^{n+r+1-p}(\log Y)) \cong R_{P,\sigma}(K_P \otimes \xi_E^{p+r+1})$$

if

(i) $H^{p+r-i}(P, \Omega_P^{n+r-p+i+1} \otimes \xi_E^{i+1}) = 0$ for all $0 \leq i \leq p+r-1$

(ii) $H^{p+r-i}(P, \Omega_P^{n+r-p+i+1} \otimes \xi_E^{i}) = 0$ for all $1 \leq i \leq p+r-1$

(iii) $H^{p+r-i}(P, \Omega_P^{n+r-p+i} \otimes \xi_E^{i}) = 0$ for all $1 \leq i \leq p+r-1$.

Since the computations for working out the conditions (i)–(iii) are similar in each case, we only treat the condition (iii). By Lemma 1.3.5 it suffices to show that

$$H^{p+r-i+t}(P, \pi^*(\Omega_Y^e \otimes \bigwedge^{f+t+1} E) \otimes \xi_E^{i-f-t-1}) = 0$$

for all $e \geq 0$ and $f \geq 0$ such that $e + f = n + r - p + i$ and $f + t \leq r$. Since $r - f \geq t$, it follows that $e = (n+r-p+i+1) - f \geq n-p+i+t+1$. Hence

$$(p+r-i+t) + e \geq (p+r-i+t) + (n-p+i+t+1) = n+r+2t+1,$$

and if $t \geq 1$ we are done by the Kodaira–Nakano vanishing theorem. As $(p+r-i) + e = n + r + (r-f)$, we find that $0 \leq r - f \leq 1$; we work out the remaining cases

1. $t = 0$, $e = n - p + i$, $f = r$
2. $t = 0$, $e = n - p + i + 1$, $f = r - 1$

to obtain the conditions (3) and (4): in Case 1 we have to show that

$$H^{p+r-i}(P, \pi^*(\Omega_Y^{n-p+i} \otimes \det E) \otimes \xi_E^{i-r-1}) = 0.$$

Set $\nu = i - r - 1$. If $-r-1 < \nu < 0$ then we are done, since $\pi_* \xi_E^{\nu} = 0$. If $0 \leq \nu \leq p-2$ we obtain condition (3). In a similar way we show that in Case 2 we obtain condition (4) and that (i) and (ii) give rise to conditions (1) and (2). $\square$

Remark 1.3.8. The Jacobi ring R carries an obvious bigrading given by

$$R_{p,q} = R_{P,\sigma}(K_P^q \otimes \xi_E^{p+1}).$$

In the case of projective space, it is more natural to grade the rings S and R by the Picard group of P; see Chapter 2. Therefore we have chosen to follow this convention throughout the thesis. If $\operatorname{Pic} Y = \mathbb{Z}$ we set

$$\begin{aligned} d(X) &= \deg(K_Y) + \sum_{i=0}^{r} d_i \\ &= \deg(K_Y \otimes \det E) \end{aligned}$$

and define a bigrading on S and R by

$$\begin{aligned} S_{p,q.d(X)} &= H^0(P, K_P^{\otimes q} \otimes \xi_E^{p+q(r+1)}) \\ &\cong H^0(Y, (K_Y \otimes \det E)^{\otimes q} \otimes S^p E) \\ R_{p,q.d(X)} &= R_{P,\sigma}(K_P^{\otimes q} \otimes \xi_E^{p+q(r+1)}). \end{aligned}$$

Corollary 1.3.9. *If the conditions (1)–(4) of Lemma 1.3.7 are satisfied, we have an exact sequence*

$$0 \to H_{\mathrm{pr}}^{n-p+1,p}(Y) \to R_{p,d(X)} \to H_{\mathrm{var}}^{n-p,p}(X) \to 0.$$

Proof: Apply Lemmas 1.3.4 and 1.3.7 to the short exact sequence

$$0 \to H_{\mathrm{pr}}^{p+r}(\Omega_P^{n-p+r+1}) \to H^{p+r}(\Omega_P^{n-p+r+1}(\log \mathcal{X})) \to H_{\mathrm{var}}^{p+r}(\Omega_{\mathcal{X}}^{n-p+r}) \to 0.$$

$\square$

Chapter 2

Complete intersections in projective space

2.1 Introduction

In an attempt to generalize the classical Noether–Lefschetz theorem for surfaces of degree $d \geq 4$ in $\mathbb{P}^3$, Griffiths and Harris [GH2] raised a number of questions concerning the behaviour of curves on a very general threefold X of degree $d \geq 6$ in $\mathbb{P}^4$. One of their questions is whether the image of the Abel–Jacobi map $\psi_X : \mathrm{CH}^2_{\mathrm{hom}}(X) \to J^2(X)$ is zero. Green [G3] and Voisin [V2] partially solved this problem; they showed that the image of ψ_X is contained in the torsion points of $J^2(X)$. A similar statement holds for odd-dimensional hypersurfaces in projective space: if $X = V(d) \subset \mathbb{P}^{2m}$ $(m \geq 2)$ is a very general hypersurface of degree $d \geq 4 + 2/(m-1)$, then the image of the Abel–Jacobi map ψ_X is contained in the torsion points of $J^m(X)$; see [G3].

We extend the result of Green and Voisin to smooth complete intersections of odd dimension in projective space (Theorem 2.4.1). In all but one of the cases where the conditions of Theorem 2.4.1 are not satisfied, it is known that the image of the Abel–Jacobi map is indeed non–torsion for a very general member of the family of complete intersections under consideration. We shall deal with the remaining exceptional case, the case of complete intersections of four quadrics, in Chapter 3.

To extend the result of Green and Voisin, we have to find an efficient algebraic description of the variable cohomology of complete intersections, analogous to the Jacobi ring description in the case of projective hypersurfaces. This problem has been solved through the work of various people, including Terasoma [Te2], Konno [Ko2], Libgober–Teitelbaum [Li], [LT] and

Dimca [Di2]; see also [ENS] and [BB]. The starting point is the following observation, due to Terasoma: if $X = V(f_0, \ldots, f_r)$ is a smooth complete intersection of multidegree $(d_0, \ldots, d_r)$ in $\mathbb{P}^{n+r+1}$ with $d_0 = \ldots = d_r = d$, then the variable cohomology of X is isomorphic (up to a shift in the Hodge filtration) to the variable cohomology of the hypersurface $\mathcal{X} = V(F) \subset \mathbb{P}^r \times \mathbb{P}^{n+r+1}$ of type $(1, d)$ defined by the bihomogeneous polynomial

$$F(x, y) = y_0 f_0(x) + \ldots + y_r f_r(x).$$

Konno extended this approach to the case of arbitrary multidegree by viewing $(f_0, \ldots, f_r)$ as a section of the vector bundle $E = \mathcal{O}_{\mathbb{P}}(d_0) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}}(d_r)$. The product of projective spaces is replaced by the projective bundle $\mathbb{P}(E^\vee)$, and $\mathcal{X}$ is replaced by the zero locus of the associated section of the tautological line bundle ξ_E on $\mathbb{P}(E^\vee)$. The variable cohomology is then described via the Jacobi ring that was introduced in Chapter 1. Working with an additional hypothesis, Libgober obtained a description of the variable cohomology via residues of differential forms defined on $\mathbb{P}^{n+r+1}$, in the spirit of the work of Griffiths [Gr2]; he observes that the variable cohomology is related to a quotient of the ring $S = \mathbb{C}[x_0, \ldots, x_{n+r+1}, y_0, \ldots, y_r]$, where S carries a suitable bigrading.

These different approaches were elegantly combined in the recent work of Dimca. He observes that $\mathbb{P}(E^\vee)$, being a smooth and compact toric variety, can be constructed as a geometric quotient. This explains the bigrading on S and shows that $\mathbb{P}(E^\vee)$ behaves like the ordinary projective space in many ways. Given this, Terasoma's original approach goes through with only minor modifications. See also [CCD] and the recent survey paper on toric geometry by D. Cox [Cox2], where the trick of passing to the hypersurface $\mathcal{X} \subset \mathbb{P}(E^\vee)$ is baptized the Cayley trick.

In Section 2.2, Dimca's method is used to give a description of the variable cohomology in terms of the Jacobi ring of $\mathcal{X}$ in $\mathbb{P}(E^\vee)$. We have given a down–to–earth presentation based on toric geometry, altough this leads to some overlap with the results of Chapter 1. Next we discuss the so–called symmetrizer lemma in Section 2.3, using which we prove our main result in Section 2.4. This chapter is a revised version of the paper [Na].

2.2 Description of the Jacobi ring

Let $X = V(d_0, \ldots, d_r) \subset \mathbb{P}^{n+r+1}$ be a smooth complete intersection of dimension $n \geq 3$, where $d_i \geq 2$ for $i = 0, \ldots, r$. We assume for the moment that $r > 0$, i.e., X is not a hypersurface. The $(r + 1)$–tuple $(f_0, \ldots, f_r)$ of equations that define X represents a global section of the vector bundle $E = \mathcal{O}_{\mathbb{P}}(d_0) \oplus \ldots \mathcal{O}_{\mathbb{P}}(d_r)$. Let $P = \mathbb{P}(E^\vee)$ be the projective bundle whose fiber over a point $x \in \mathbb{P}^{n+r+1}$ is the projective space of hyperplanes in E_x. Using the results in [Cox1], we can associate to the smooth and compact toric variety P its 'homogeneous coordinate ring' $S = \mathbb{C}[x_0, \ldots, x_{n+r+1}, y_0, \ldots, y_r]$. This ring carries a natural grading by elements of $\mathrm{Pic}(P) = \mathrm{Pic}(\mathbb{P}^{n+r+1}) \times \mathbb{Z} \cong \mathbb{Z}^2$. The variables x_i $(i = 0, \ldots, n+r+1)$ have bidegree $(0,1)$; the variables y_j $(j = 0, \ldots, r)$ have bidegree $(1, -d_j)$. Set

$$F(x,y) = y_0 f_0(x) + \ldots + y_r f_r(x),$$

and let $\mathcal{X} \subset P$ be the divisor defined by $F(x,y) \in S_{1,0}$.

Remark 2.2.1. Set $N = n + r + 1$. We denote the open subset $\mathbb{C}^N \setminus \{0\} \times \mathbb{C}^{r+1} \setminus \{0\} \subset \mathbb{C}^{N+r+1}$ by U. The presence of a bigrading on the ring S is a consequence of the construction of P as a geometric quotient U/G, where $G = \mathbb{C}^* \times \mathbb{C}^*$ acts on U by

$$(t_1, t_2).(x_0, \ldots, x_N, y_0, \ldots, y_r) = (t_2 x_0, \ldots, t_2 x_N, t_2^{-d_0} t_1 y_0, \ldots, t_2^{-d_r} t_1 y_r).$$

Good references for this construction are [Cox1], [Bat] and [BC].

Because $F(x,y) \in S_{1,0}$ represents the global section of the tautological line bundle $\xi_E = \mathcal{O}_P(1)$ that corresponds to $(f_0, \ldots, f_r)$ under the canonical isomorphism $H^0(P, \xi_E) = H^0(\mathbb{P}^n, E)$, Lemma 1.3.1 shows that $\mathcal{X} \subset P$ is a very ample divisor if and only if $d_i > 0$ for all $i = 0, \ldots, r$.

Note that X is non–singular if and only if $\mathcal{X}$ is non–singular (see Lemma 1.3.2). Let i (resp. j) be the inclusion of X in $\mathbb{P}^{n+r+1}$ (resp. the inclusion of $\mathcal{X}$ in P). As in Chapter 1, we define the *variable cohomology* of X and $\mathcal{X}$ as

$$\begin{aligned} H^n_{\mathrm{var}}(X, \mathbb{Q}) &= \ker i_* : H^n(X, \mathbb{Q}) \to H^{n+2r+2}(\mathbb{P}^{n+r+1}, \mathbb{Q}) \\ H^{n+2r}_{\mathrm{var}}(\mathcal{X}, \mathbb{Q}) &= \ker j_* : H^{n+2r}(\mathcal{X}, \mathbb{Q}) \to H^{n+2r+2}(P, \mathbb{Q}). \end{aligned}$$

The notions of variable and primitive cohomology are strongly related; see Lemma 1.2.5. Let $\pi : P \to \mathbb{P}^n$ be the projection, and set $\tilde{X} = \pi^{-1}(X) = X \times \mathbb{P}^r$. By Lemma 1.3.4 the inclusion of $\tilde{X}$ in $\mathcal{X}$ induces an isomorphism of Hodge structures

$$H^{n+2r}_{\mathrm{var}}(\mathcal{X}, \mathbb{C}) \xrightarrow{\sim} H^n_{\mathrm{var}}(X, \mathbb{C}) \otimes H^{2r}(\mathbb{P}^r, \mathbb{C}).$$

The description of the variable cohomology of $\mathcal{X}$ strongly resembles the description of the primitive cohomology of a hypersurface in $\mathbb{P}^{n+1}$. As most of the results are similar to those in [Di1], [Do] and [CGGH], we shall omit their proofs.

The Euler vector fields

$$e_1 = \sum_{i=0}^{r} y_i \frac{\partial}{\partial y_i}$$

and

$$e_2 = \sum_{i=0}^{n+r+1} x_i \frac{\partial}{\partial x_i} - \sum_{i=0}^{r} d_i y_i \frac{\partial}{\partial y_i}$$

generate the action of $G = \mathbb{C}^* \times \mathbb{C}^*$ on $U = \mathbb{C}^{n+r+1} \setminus \{0\} \times \mathbb{C}^{r+1} \setminus \{0\}$. The orbits of this action are the fibers of $\pi : U \to P$.

Definition 2.2.2.

(i) *For a monomial* $f = x_0^{\alpha_0} \dots x_N^{\alpha_N} y_0^{\beta_0} \dots y_r^{\beta_r}$ *we define* $|f|_1 = \sum_{i=0}^{r} \beta_i$, $|f|_{2,x} = \sum_{i=0}^{n+r+1} \alpha_i$, $|f|_{2,y} = -\sum_{j=0}^{r} d_j \beta_j$ *and* $|f|_2 = |f|_{2,x} + |f|_{2,y}$.

(ii) *For a differential form* $\omega = f.dx_{s_1} \wedge \dots \wedge dx_{s_i} \wedge dy_{t_1} \wedge \dots \wedge dy_{t_j}$ *we set* $|\omega|_1 = |f|_1 + j$, $|\omega|_{2,x} = |f|_{2,x} + i$, $|\omega|_{2,y} = |f|_{2,y} - \sum_{k=1}^{j} d_{t_k}$ *and* $|\omega|_2 = |\omega|_{2,x} + |\omega|_{2,y}$.

In the statement of the following Lemma, the differential d is written in the form $d = d_x + d_y$. We write $A^k = H^0(\mathbb{C}^{n+2r+2}, \Omega^k_{\mathbb{C}^{n+2r+2}})$, and denote the contraction with a vector field e by i_e.

Lemma 2.2.3. *If* $\omega \in A^i$ *and* $\omega' \in A^j$, *then*

(i) $i_e(\omega \wedge \omega') = i_e(\omega) \wedge \omega' + (-1)^i \omega \wedge i_e(\omega')$

(ii) $i_{e_1}(df) = |f|_1 f$, $i_{e_2}(df) = |f|_2 f$

(iii) $d_y(i_{e_1}\omega) + i_{e_1}(d_y\omega) = |\omega|_1\, \omega$

(iv) $d_x(i_{e_2}\omega) + i_{e_2}(d_x\omega) = |\omega|_{2,x}\, \omega$

(v) $d_y(i_{e_2}\omega) + i_{e_2}(d_y\omega) = |\omega|_{2,y}\, \omega$

(vi) $|i_{e_1}(\omega)|_k = |i_{e_2}(\omega)|_k = |\omega|_k$, $k = 1, 2$.

Lemma 2.2.4. *A rational* k*–form* φ *on* U *given by*

$$\varphi = \frac{1}{H(x,y)} \sum_{\substack{I,J \\ |I|+|J|=k}} A_{I,J}(x,y)\, dx_I \wedge dy_J$$

satisfies $\varphi = \pi^*\omega$ *for a rational* k*–form* ω *on* P *if and only if*

(i) φ *is* G*–invariant, i.e.,* $|\varphi|_1 = |\varphi|_2 = 0$.

(ii) $i_{e_1}(\varphi) = i_{e_2}(\varphi) = 0$.

Proof: One easily checks that $\varphi = \pi^*\omega$ for a rational k–form ω on P if and only if φ and $d\varphi$ are horizontal, i.e., $i_e(\varphi) = i_e(d\varphi) = 0$ for all vertical vector fields e. This is equivalent to $i_{e_1}(\varphi) = i_{e_2}(\varphi) = i_{e_1}(d\varphi) = i_{e_2}(d\varphi) = 0$, hence the assertion follows from the previous Lemma. $\square$

From now on we shall identify rational differential forms on P with their pullbacks to U.

Lemma 2.2.5. *Suppose that* $\psi \in A^k$ *satisfies the following conditions*

(i) $i_{e_1}(\psi) = i_{e_2}(\psi) = 0$

(ii) $|\psi|_1 \neq 0$ *and at least one of* $|\psi|_2$, $|\psi|_{2,x}$ *is nonzero.*

Then $\psi = i_{e_2} i_{e_1}(\varphi)$ *for some* $\varphi \in A^{k+2}$.

Proof: If $|\psi|_1 = \alpha \neq 0$ and $|\psi|_{2,x} = \beta \neq 0$, we can write

$$\begin{aligned} \alpha\beta\psi &= \alpha(i_{e_2}(d_x\psi) + d_x(i_{e_2}\psi)) \\ &= \alpha i_{e_2} d_x \psi = i_{e_2}(d_x(\alpha\psi)) \\ &= i_{e_2}(d_x(i_{e_1}(d_y\psi) + d_y(i_{e_1}\psi))) \\ &= i_{e_2}(d_x(i_{e_1}(d_y\psi))) = -i_{e_2} i_{e_1}(d_x d_y \psi). \end{aligned}$$

$\square$

Corollary 2.2.6.

$$\psi \in H^0(P, \Omega^k_P(q\mathcal{X})) \iff \psi = \frac{i_{e_2} i_{e_1}(\varphi)}{F(x,y)^q}$$

for some $\varphi \in A^{k+2}$ *with* $|\varphi|_1 = q$ *and* $|\varphi|_2 = 0$.

The following Lemma shows how to express $d\psi$ in a similar form:

Lemma 2.2.7. *If*

$$\psi = \frac{i_{e_2} i_{e_1}(\varphi)}{F^q},$$

then

$$d\psi = \frac{i_{e_2} i_{e_1}(F d\varphi - q\, dF \wedge \varphi)}{F^{q+1}}.$$

Lemma 2.2.8.

(i)

$$\psi \in H^0(P, \Omega_P^{n+2r+1}((q+1)\mathcal{X})) \iff \psi = \frac{P(x,y)\Omega}{F^{q+1}}$$

where $\Omega = i_{e_2} i_{e_1}(dx_0 \wedge \ldots \wedge dx_{n+r+1} \wedge dy_0 \wedge \ldots \wedge dy_r)$.

(ii)

$$\tilde{\psi} \in H^0(P, \Omega_P^{n+2r}(q\mathcal{X}) \iff \tilde{\psi} = \frac{i_{e_2} i_{e_1}(\varphi)}{F^q}$$

where

$$\varphi = \sum_{i=0}^{n+r+1} Q_i(x,y)\Omega_i \wedge dy_0 \wedge \ldots \wedge dy_r + \sum_{\alpha=0}^{r} R_\alpha(x,y) dx_0 \wedge \ldots \wedge dx_n \wedge \Omega_\alpha,$$

$$\Omega_i = dx_0 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_{n+r+1},$$
$$\Omega_\alpha = dy_0 \wedge \ldots \wedge \widehat{dy_\alpha} \wedge \ldots \wedge dy_r.$$

Definition 2.2.9. *The Jacobi ideal $J(F) \subset S$ is the ideal in S generated by the partial derivatives*

$$\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_{n+r+1}}, \frac{\partial F}{\partial y_0}, \ldots, \frac{\partial F}{\partial y_r}.$$

The Jacobi ring R is the quotient ring $S/J(F)$. The bigrading on S induces a bigrading on R.

Proposition 2.2.10. *There is a natural isomorphism*

$$H_{\mathrm{var}}^{n-p,p}(X) \cong R_{p,d(X)},$$

where $n = \dim\, X$ and $d(X) = \sum_{i=0}^{r} d_i - n - r - 2$.

Proof: We have already seen that $H_{\mathrm{var}}^{n-p,p}(X) \cong H_{\mathrm{var}}^{n-p+r,p+r}(\mathcal{X})$. There is an exact sequence

$$0 \to H_{\mathrm{pr}}^{n-p+r+1,p+r}(P) \to H^{p+r}(\Omega_P^{n-p+r+1}(\log \mathcal{X})) \to H_{\mathrm{var}}^{n-p+r,p+r}(\mathcal{X}) \to 0.$$

From 1.3.4 we obtain $H_{\mathrm{pr}}^{n+2r+1}(P) \cong H_{\mathrm{pr}}^{n+1}(\mathbb{P}^{n+r+1}) = 0$. Hence

$$H_{\mathrm{var}}^{n-p+r,p+r}(\mathcal{X}) \cong H^{p+r}(\Omega_P^{n-p+r+1}(\log \mathcal{X})).$$

Since $\mathcal{X} \subset P$ is an ample divisor by Lemma 1.3.1, we can apply the Bott vanishing theorem on P (see [BC, Theorem 7.1]) to obtain

$$H^i(P, \Omega_P^j(k\mathcal{X})) = 0 \ \text{ for all } i > 0, k > 0 \text{ and } j \geq 0.$$

A spectral sequence argument shows that

$$H^{p+r}(\Omega_P^{n-p+r+1}(\log \mathcal{X})) \cong \frac{H^0(\Omega_P^d((p+r+1)\mathcal{X}))}{H^0(\Omega_P^d((p+r)\mathcal{X})) + dH^0(\Omega_P^{d-1}((p+r)\mathcal{X})),}$$

where $d = \dim\ P = n + 2r + 1$.

By Lemma 2.2.8, an element of $H^0(P, \Omega_P^d((p+r+1)\mathcal{X}))$ can be written in the form

$$\psi_P = \frac{P(x,y)\Omega}{F^{p+r+1}}$$

where $\deg \Omega = (r+1, -d(X))$ and $\deg(P(x,y)) = (p, d(X))$. What we have shown so far is that the map

$$\begin{array}{rcl} \mathrm{Res} : S_{p,d(X)} & \to & F^{d-p+r} H_{\mathrm{var}}^{n+2r+1}(\mathcal{X}) \\ P(x,y) & \mapsto & [\mathrm{Res}(\psi_P)] \end{array}$$

is surjective. If $\tilde{\psi} \in H^0(P, \Omega_P^{n+2r}((p+r)\mathcal{X}))$, then $\tilde{\psi} = \frac{i_{e_2} i_{e_1}(\varphi)}{F^{p+r}}$ and

$$d\tilde{\psi} = \frac{\{F(\sum_{i=0}^N \frac{\partial Q_i}{\partial x_i} + \sum_{j=0}^r \frac{\partial R_j}{\partial y_j}) - (p+r)(\sum_{i=0}^N \frac{\partial F}{\partial x_i} Q_i + \sum_{j=0}^r \frac{\partial F}{\partial y_j} R_j\}\Omega}{F^{p+r+1}},$$

where $N = n + r + 1$. Hence

$$\psi_P \equiv d\tilde{\psi} \bmod H^0(\Omega_P^{n+2r+1}((p+r)\mathcal{X}) \iff P \in J(F).$$

This shows that Res induces an isomorphism $R_{p,d(X)} \cong H_{\mathrm{var}}^{n-p+r,p+r}(\mathcal{X})$, as desired. $\square$

Remark 2.2.11.

(i) If $r = 0$, then $\mathbb{P}(E^\vee) \cong \mathbb{P}^{n+1}$, $S = \mathbb{C}[x_0, \ldots, x_{n+1}, y_0]$ and $F(x,y) = y_0 f_0(x)$. Clearly the ring $R(F)$ is different from the Jacobi ring $R(f_0)$ of the hypersurface $V(f_0) \subset \mathbb{P}^{n+1}$, but the map $\alpha : S \to \mathbb{C}[x_0, \ldots, x_{n+1}]$ that sends $G(x_0, \ldots, x_{n+1}, y_0)$ to $G(x_0, \ldots, x_{n+1}, 1)$ induces an isomorphism $R(F)_{p,d(X)} \xrightarrow{\sim} R(f_0)_{(p+1)d_0 - n - 1}$ between the graded pieces of these rings that describe $H_{\mathrm{var}}^{n-p,p}(X)$.

(ii) The toric description of P shows that the bidegree of the canonical bundle K_P is $(-r-1, d(X))$ and $S_{p,d(X)} \cong H^0(P, K_P \otimes \xi_E^{p+r+1})$. The proof of Proposition 2.2.10 shows that the Jacobi ring $J(F)$ coincides with the Jacobi ring J defined in Chapter 1.

(iii) The description of the variable cohomology $H^n_{\mathrm{var}}(X)$ for a complete intersection X in an arbitrary smooth and compact toric variety $\mathbb{P}_\Sigma$ proceeds along the same lines. The number of Euler vector fields equals the rank of $\mathrm{Pic}(\mathbb{P}(E^\vee)) = \mathrm{Pic}(\mathbb{P}_\Sigma) \times \mathbb{Z}$.

2.3 Symmetrizer lemma

Using a version of the symmetrizer lemma, we prove that the infinitesimal invariants associated to certain normal functions are zero. Consequently these normal functions are torsion sections of the fiber space of intermediate Jacobians.

We keep the notation of Section 2.2, but from now on we consider the case where X is a smooth complete intersection in $\mathbb{P}^{2m+r}$ of odd dimension $n = 2m-1$. In this case we have $H^{2m-1}(X) = H^{2m-1}_{\mathrm{var}}(X) = H^{2m-1}_{\mathrm{pr}}(X)$. Let $U \subset \mathbb{P}H^0(\mathbb{P}^{2m+r}, E)$ be the open subset parametrizing smooth complete intersections, and let $f : X_U \to U$ be the universal family. The cohomology groups of the fibers of f give rise to a local system $H_{\mathbb{Z}} = R^{2m-1}f_*\mathbb{Z}$. Let $\mathcal{H}^{2m-1} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_U$ be the associated Hodge bundle; it is filtered by holomorphic subbundles $\mathcal{F}^p$ $(0 \leq p \leq 2m-1)$. The Hodge bundle comes equipped with a flat connection ∇, the Gauss–Manin connection, whose flat sections are the sections of the local system $H_{\mathbb{Z}}$. The filtration of subbundles $\mathcal{F}^\bullet$ is shifted by ∇ according to the Griffiths transversality rule $\nabla \mathcal{F}^p \subset \Omega^1_U \otimes \mathcal{F}^{p-1}$. Let

$$\mathcal{J}^m = \mathcal{H}^{2m-1}/(\mathcal{F}^m + H_{\mathbb{Z}})$$

be the sheaf of intermediate Jacobians over U. The Gauss–Manin connection induces a map

$$\overline{\nabla} : \mathcal{J}^m \to \Omega^1_U \otimes \mathcal{H}^{2m-1}/\mathcal{F}^{m-1},$$

whose kernel is denoted by $\mathcal{J}^m_h$. By abuse of language, a global section of $\mathcal{J}^m_h$ is called a normal function.

To study the image of the Abel–Jacobi map on X_0 using normal functions, we 'spread out' cycles on a very general fiber $X_{u_0} = f^{-1}(u_0)$ to relative cycles.

We shall briefly describe this process. The relative Hilbert functor $Hilb_{X_U/U}$ is represented by a countable union of projective schemes

$$\mathrm{Hilb}_{X_U/U} = \coprod_P \mathrm{Hilb}^P_{X_U/U};$$

see [Koll, Chapter I] or [MFK, Chapter 0, §5c]. The relative Hilbert scheme $\mathrm{Hilb}_{X_U/U}$ admits a dominant structure morphism $\mathrm{Hilb}_{X_U/U} \to U$. There exists a countable union $\{H_\alpha\}_{\alpha\in A}$ of irreducible components of $\mathrm{Hilb}_{X_U/U}$ that do not dominate U. The images of these components under the morphism $\mathrm{Hilb}_{X_U/U} \to U$ form a countable union $\{S_\alpha\}_{\alpha\in A}$ of proper subvarieties of U. Choose a point $u_0 \in U \setminus \cup_{\alpha\in A} S_\alpha$, and let $Z_0 \in \mathrm{Hilb}^P_{X_{u_0}}$. Let $\mathcal{H}_0$ be the irreducible component of $\mathrm{Hilb}_{X_U/U}$ that contains Z_0. There exists a multi–valued section Z_U of $\mathcal{H}_0$ through Z_{u_0}. Let $T \to U$ be a finite covering such that $\mathcal{H}_0$ admits a section over T (for instance, if we let T be the image of Z_U in $\mathcal{H}_0$, the diagonal $\Delta_T : T \to T \times_U T$ gives the desired section). By abuse of language, we shall also write $g : T \to U$ for the finite étale covering that is obtained by removing the branch locus.

Thus if $X_0 = V(d_0, \ldots, d_r) \subset Y$ is a very general complete intersection and $Z_0 \in Z^m_{\mathrm{hom}}(X_0)$, there exist a finite étale covering $g : T \to U$, a relative cycle $Z_T \in CH^m_{\mathrm{hom}}(X_T/T)$ and a point $t_0 \in g^{-1}(0)$ such that the fiber of Z_T over t_0 is Z_0. Set $Z_t = Z_T \cap f_T^{-1}(t)$ and let $\nu \in H^0(T, \mathcal{J}^m_h)$ be the normal function given by $\nu(t) = \psi_{X_t}(Z_t)$. Set $H_\mathbb{Q} = H_\mathbb{Z} \otimes_Z \mathbb{Q}$. The twisted De Rham complex

$$\Omega^\bullet_T \otimes H_\mathbb{Q} : \mathcal{H} \to \Omega^1_T \otimes \mathcal{H} \to \Omega^2_T \otimes \mathcal{H} \to \cdots$$

is a resolution of the local system $H_\mathbb{C} = H_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{C}$. Let $F^m(\Omega^\bullet_T \otimes H_\mathbb{Q})$ be the subcomplex

$$\mathcal{F}^m \to \Omega^1_T \otimes \mathcal{F}^{m-1} \to \Omega^2_T \otimes \mathcal{F}^{m-2} \to \cdots$$

of $\Omega^\bullet_T \otimes H_\mathbb{Q}$. Note that although the differential ∇ of $F^m(\Omega^\bullet_T \otimes H_\mathbb{Q})$ is not $\mathcal{O}_T$–linear, the induced differential $\overline{\nabla}$ on the graded pieces $\mathrm{Gr}^p_F(\Omega^\bullet_T \otimes H_\mathbb{Q})$ is $\mathcal{O}_T$–linear. The normal function ν has an infinitesimal invariant $\delta\nu \in H^0(T, \mathcal{H}^1(F^m(\Omega^\bullet_T \otimes H_\mathbb{Q})))$; see [G4, Lecture 6]. It is known that ν has flat local liftings if and only if $\delta_\nu = 0$ [loc. cit.]. Hence, to prove that $\delta\nu = 0$ it suffices to show that

$$\mathcal{H}^1(\mathrm{Gr}^p_F(\Omega^\bullet_T \otimes H_\mathbb{Q})) = 0 \text{ for all } p \geq m.$$

Let T_0 be the tangent space to U at $0 \in U$. As $g : T \to U$ is an étale covering, the tangent space to T at $t_0 \in g^{-1}(0)$ is isomorphic to T_0. We want to show that the first cohomology group of the complex

$$H^{p,2m-p-1}(X_0) \to T_0^\vee \otimes H^{p-1,2m-p}(X_0) \to \bigwedge^2 T_0^\vee \otimes H^{p-2,2m-p+1}(X_0)$$

vanishes. Dualizing this complex we obtain

$$\bigwedge^2 T_0 \otimes H^{2m-p+1,p-2}(X_0) \to T_0 \otimes H^{2m-p,p-1}(X_0) \to H^{2m-p-1,p}(X_0).$$

Lemma 2.3.1. *The diagram*

$$\begin{array}{ccccc} \bigwedge^2 S_{1,0} \otimes R_{p-2,d(X)} & \to & S_{1,0} \otimes R_{p-1,d(X)} & \to & R_{p,d(X)} \\ \downarrow & & \downarrow & & \downarrow \\ \bigwedge^2 T_0 \otimes H^{2m-p+1,p-2}(X_0) & \to & T_0 \otimes H^{2m-p,p-1}(X_0) & \to & H^{2m-p-1,p}(X_0) \end{array}$$

is commutative.

Proof: This is a standard consequence of the description of $H^{2m-1}(X)$ by residues of differential forms. Note that the identification of the tangent space $T_{U,0}$ with $S_{1,0}$ is obtained by sending a polynomial $G(x,y) \in S_{1,0}$ to the infinitesimal deformation of $\mathcal{X}$ given by $F_t(x,y) = F(x,y) + t\,G(x,y)$, $t^2 = 0$. The commutativity of the right square is established by the following basic observation: if we write

$$\Omega_{P(t)} = \frac{P(t)\Omega}{(F+tG)^{p+r}},$$

then

$$\partial/\partial t\, \Omega_{P(t)}|_{t=0} \equiv -(p+r)\frac{P(0)G\Omega}{F^{p+r+1}}$$

modulo differential forms with poles of lower order. It follows that

$$\begin{aligned} \nabla_{\partial/\partial t} \operatorname{Res} \Omega_{P(t)} &= \partial/\partial t(\operatorname{Res} \Omega_{P(t)})|_{t=0} \\ &= \operatorname{Res}(\partial/\partial t\, \Omega_{P(t)}|_{t=0}) = \operatorname{Res}(\Omega_{PG}). \end{aligned}$$

The commutativity of the square on the left hand side follows in a similar way. □

In the sequel we shall use some standard multi–index notation. For a multi–index $I = (i_0, \ldots, i_r)$ we write $\langle d, I\rangle = d_0 i_0 + \ldots + d_r i_r$. Let (i_0) denote the $(r+1)$–tuple $(0,\ldots,0,1,0,\ldots,0)$ where the number 1 occurs at position i_0.

Lemma 2.3.2. *The multiplication map*

$$S_{a,b} \otimes S_{\alpha,\beta} \longrightarrow S_{a+\alpha,b+\beta}$$

is surjective if

(i) $a \geq 0$, $\alpha \geq 0$

(ii) $\langle d, I\rangle + b \geq 0$ *for all* I *with* $|I| = a$, $\langle d, J\rangle + \beta \geq 0$ *for all* J *with* $|J| = \alpha$.

Proof: Note that $S_{a,b}$ is spanned by the monomials $y^I x^J$ with $|I| = a$, $|J| = \langle d, I\rangle + b$. Given a monomial $x^K y^L$ with $|L| = a+\alpha$, $|K| = \langle d, L\rangle + b + \beta$ we can write $L = L_1 \cup L_2$ with $|L_1| = a$, $|L_2| = \alpha$ and $K = K_1 \cup K_2$ where $|K_1| = \langle d, L_1\rangle + b$, $|K_2| = \langle d, L_2\rangle + \beta$. □

Lemma 2.3.3. (symmetrizer lemma) *Assume that* $n \geq 2$, $p \geq 2$ *and* $d_0 \geq \ldots \geq d_r$. *The complex*

$$\bigwedge^2 S_{1,0} \otimes R_{p-2,d(X)} \to S_{1,0} \otimes R_{p-1,d(X)} \to R_{p,d(X)}$$

is exact at the middle term if the following two conditions are satisfied:

$(*)$ $d_0 + \ldots + d_r + (p-2)d_r \geq n + r + 3$

$(**)$ $d_1 + \ldots + d_r + (p-1)d_r \geq n + r + 2.$

To prove the symmetrizer lemma, it suffices to show that

(i) The complex

$$\bigwedge^2 S_{1,0} \otimes S_{p-2,d(X)} \xrightarrow{g} S_{1,0} \otimes S_{p-1,d(X)} \xrightarrow{h} S_{p,d(X)}$$

is exact at the middle term.

(ii) The map

$$S_{1,0} \otimes J_{p-1,d(X)} \to J_{p,d(X)}$$

is surjective.

This follows by chasing the commutative diagram with exact columns

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\bigwedge^2 S_{1,0} \otimes J_{p-2,d(X)} & \to & S_{1,0} \otimes J_{p-1,d(X)} & \to & J_{p,d(X)} \\
\downarrow & & \downarrow & & \downarrow \\
\bigwedge^2 S_{1,0} \otimes S_{p-2,d(X)} & \to & S_{1,0} \otimes S_{p-1,d(X)} & \to & S_{p,d(X)} \\
\downarrow & & \downarrow & & \downarrow \\
\bigwedge^2 S_{1,0} \otimes R_{p-2,d(X)} & \to & S_{1,0} \otimes R_{p-1,d(X)} & \to & R_{p,d(X)} \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0.
\end{array}$$

We shall verify the conditions (i) and (ii) in Lemmas 2.3.4 and 2.3.7.

Lemma 2.3.4. *The complex*

$$\bigwedge^2 S_{1,0} \otimes S_{p-2,k} \xrightarrow{g} S_{1,0} \otimes S_{p-1,k} \xrightarrow{h} S_{p,k}$$

is exact at the middle term provided that $p \geq 2$ and $\langle d, J\rangle + k > 0$ for all multi–indices J with $|J| = p-2$.

Proof: The map g is given by

$$g(y_{i_0}x^{I_0} \wedge y_{i_1}x^{I_1} \otimes y^K x^L) = y_{i_0}x^{I_0} \otimes y^{K+(i_1)}x^{L+I_1} - y_{i_1}x^{I_1} \otimes y^{K+(i_0)}x^{L+I_0}.$$

This shows that

$$y_{i_0}x^{I_0} \otimes y^{J_0}x^{K_0} \equiv y_{i_1}x^{I_1} \otimes y^{J_1}x^{K_1} \quad \text{mod } (\operatorname{im} g)$$

if $J_0 + (i_0) = J_1 + (i_1)$, $K_0 + I_0 = K_1 + I_1$ and $K_1 - I_0 \geq 0$. In fact, if these conditions are satisfied it follows that $K_0 - I_1 \geq 0$ and

$$\begin{aligned} M = K_0 + I_0 = K_1 + I_1 &= I_0 + I_1 + (K_0 - I_1) \\ &= I_0 + I_1 + (K_1 - I_0); \end{aligned}$$

hence $K_0 - I_1 = K_1 - I_0 = L$, $|L| = \langle d, J\rangle + k$, and $J = J_0 - (i_1) = J_1 - (i_0)$. Combining the two relations

$$\begin{aligned} y_{i_0}x^{I_0} \otimes y^{J_0}x^{M-I_0} &\equiv y_{i_1}x^{M-L} \otimes y^{J_1}x^L \\ &\equiv y_{i_2}x^{I_2} \otimes y^{J_2}x^{M-I_2}, \end{aligned}$$

we find that

$$y_{i_0}x^{I_0} \otimes y^{J_0}x^{M-I_0} \equiv y_{i_2}x^{I_2} \otimes y^{J_2}x^{M-I_2}$$

if $J_0 + (i_0) = J_2 + (i_2)$ and if there exists an L with $L \leq M$, $L - I_0 \geq 0$ and $L - I_2 \geq 0$. Here we choose J_1 and i_1 in the following way: take $i_1 = \max(i_0, i_2)$ and take $J_1 = J_\alpha$ if $i_1 = i_\alpha$, $\alpha \in \{0, 2\}$. Notice that $|L| = \langle d, J_1\rangle + k$.

If $J_0 - J_2 = (i_2) - (i_0)$ and $I_0 - I_2 = (k_0) - (k_2)$ (i.e., I_0 and I_2 also differ by one change of index), we can choose L with $L - I_0 \geq 0$ and $L - I_2 \geq 0$ if $|L| > |I_0|$ and $|L| > |I_2|$, i.e., if

$$\langle d, J_1\rangle + k > \max(d_{i_0}, d_{i_2}).$$

By construction this holds if $\langle d, J\rangle + k > 0$, where we set $J = J_1 - (i_0)$ if $i_1 = i_2$ and $J = J_1 - (i_2)$ if $i_1 = i_0$.

By transitivity we can show the existence of L if I_0 and I_2 differ by more than one change of index. Hence

$$y_{i_0}x^{I_0} \otimes y^{J_0}x^{M-I_0} \equiv y_{i_2}x^{I_2} \otimes y^{J_2}x^{M-I_2}$$

if $J_0 + (i_0) = J_2 + (i_2)$, $M \geq I_0$, $M \geq I_1$ and $\langle d, J\rangle + k > 0$ ($J = J_0 - (i_2) = J_2 - (i_0)$). If

$$\begin{aligned} h(\sum_{i,I,K,L} c_{i,I,K,L} \;\; y_i x^I \otimes y^K x^L) &= \sum_{i,I,K,L} c_{i,I,K,L} \;\; y^{K+(i)} x^{I+L} \\ &= \sum_{(J,M)} \sum_{\substack{(i,I,K,L) \\ K+(i)=J, I+L=M}} c_{i,I,K,L} \;\; y^J x^M = 0, \end{aligned}$$

then

$$\sum_{\substack{(i,I,K,L) \\ K+(i)=J, I+L=M}} c_{i,I,K,L} = 0$$

for all pairs (J, M), hence

$$\sum_{i,I,K,L} c_{i,I,K,L} \;\; y_i x^I \otimes y^K x^L \equiv 0 \mod (\operatorname{im} g).$$

$\square$

Remark 2.3.5. The proof of Lemma 2.3.4 is based on the proof of the symmetrizer lemma for projective hypersurfaces by Donagi and Green [DG]. It is possible to prove Lemmas 2.3.4 and 2.3.7 in a different way, using the abstract definition of the Jacobi ring from Chapter 1 and Castelnuovo–Mumford regularity.

Corollary 2.3.6. *Suppose that $n \geq 2$ and $p \geq 2$. The complex*

$$\bigwedge^2 S_{1,0} \otimes S_{p-2,d(X)} \xrightarrow{g} S_{1,0} \otimes S_{p-1,d(X)} \xrightarrow{h} S_{p,d(X)}$$

is exact at the middle term if condition $(*)$ of Lemma 2.3.3 is satisfied.

Proof: Apply Lemma 2.3.4 with $k = d(X)$. $\square$

Lemma 2.3.7. *Suppose that $n \geq 2$ and $p \geq 2$. If the conditions $(*)$ and $(**)$ of Lemma 2.3.3 are satisfied, the map*

$$S_{1,0} \otimes J_{p-1,d(X)} \to J_{p,d(X)}$$

is surjective.

Proof: As $\deg(\partial F/\partial x_k) = (1,-1)$ $(0 \le k \le 2m+r)$ and $\deg(\partial F/\partial y_i) = (0,d_i)$ $(0 \le i \le r)$, it suffices to show that the map μ' that appears in the commutative diagram

$$\begin{array}{ccc} S_{1,0} \otimes S_{p-a-1,d(X)-b} \otimes J_{a,b} & \xrightarrow{\mu'} & S_{p-a,d(X)-b} \otimes J_{a,b} \\ \downarrow \nu & & \downarrow \nu' \\ S_{1,0} \otimes J_{p-1,d(X)} & \xrightarrow{\mu} & J_{p,d(X)} \end{array}$$

is surjective if $(a,b) = (1,-1)$ and if $(a,b) = (0,d_i)$ $(i = 0,\ldots,r)$. If $(a,b) = (1,-1)$, then μ' is surjective if $\langle d,J\rangle + d(X) + 1 \ge 0$ for all J with $|J| = p-2$; this follows from condition $(*)$.

If $(a,b) = (0,d_i)$ $(i = 0,\ldots,r)$, then μ' is surjective if

$$\langle d,J\rangle + d(X) - d_i \ge 0$$

for all J with $|J| = p-1$. This follows from the condition $(**)$. □

Corollary 2.3.8. *Suppose that $m \ge 2$ and $d_0 \ge \ldots \ge d_r$. Then $\delta\nu = 0$ provided that*

(1) $d_0 + \ldots + d_r + (m-2)d_r \ge 2m + r + 2$

(2) $d_1 + \ldots + d_r + (m-1)d_r \ge 2m + r + 1.$

Proof: This follows from Lemmas 2.3.1 and 2.3.3. □

Note that the first condition in Corollary 2.3.8 is implied by the second one, unless $d_0 = \ldots = d_r$.

Lemma 2.3.9. *If the conditions* (1) *and* (2) *of Corollary 2.3.8 are satisfied, the normal function ν has flat local liftings that are unique up to sections of $H_{\mathbb{Z}}$.*

Proof: It follows from (1) and (2) that $\delta\nu = 0$, hence ν has flat local liftings. If $\tilde{\nu}$ and $\tilde{\nu}'$ are two flat liftings of ν over an open set $U_0 \subset T$, we can write

$$\tilde{\nu} - \tilde{\nu}' = \varphi + \lambda$$

where $\varphi \in H^0(U_0, \mathcal{F}^m)$ and $\lambda \in H^0(U_0, H_{\mathbb{Z}})$. Since $\nabla\varphi = \nabla(\tilde{\nu} - \tilde{\nu}') = 0$, it suffices to show that the map

$$\nabla : \mathcal{F}^m \to \Omega^1_T \otimes \mathcal{F}^{m-1}$$

is injective. By duality, it suffices to show that for all $t \in T$ the map

$$T \otimes H^{2m-p,p-1}(X_t) \to H^{2m-p-1,p}(X_t)$$

is surjective for $p \geq m$. This follows if the map

$$S_{1,0} \otimes S_{p-1,d(X)} \longrightarrow S_{p,d(X)}$$

is surjective, and by Lemma 2.3.2 this holds if $\langle d, J \rangle + d(X) > 0$ for all J with $|J| = p - 1$. This follows from condition (1). $\square$

If the conditions (1) and (2) of Corollary 2.3.8 are satisfied, then the normal function ν is torsion. This is proved using a monodromy argument, which is taken from [V4, Lecture 4].

Lemma 2.3.10. *If ν has flat local liftings that are unique up to sections of $H_{\mathbb{Z}}$, then $\nu \in H^0(T, \mathcal{J}_h^m)$ is a torsion section of $\mathcal{J}$.*

Proof: Let $\tilde{\nu}$ be a flat local lifting of ν in an open neighbourhood of $t_0 \in T$. We have to show that $\tilde{\nu}(t_0) \in H^{2m-1}(X_0, \mathbb{Q})$. To this end we take a loop $\gamma : [0,1] \to T$ based at t_0 and cover it by simply connected open sets U_α ($\alpha = 1, \ldots, k$) such that ν has a flat lifting ν_α on U_α. For all α, $\beta \in \{1, \ldots, k\}$ we have $\nu_\alpha - \nu_\beta = \lambda_{\alpha\beta}$ for some $\lambda_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, H_{\mathbb{Z}})$; hence we can modify ν_2 by λ_{12} to obtain $\nu_1 = \nu_2$ on $U_1 \cap U_2$. Proceeding in this way on $U_2 \cap U_3, \ldots, U_{k-1} \cap U_k$, we find a new flat lifting $\hat{\nu}$ of ν in $\gamma(1)$. Let $\rho : \pi_1(T, t_0) \to \operatorname{Aut} H^{2m-1}(X_0, \mathbb{C})$ be the monodromy representation. By definition we have

$$\rho(\gamma)(\tilde{\nu}(t_0)) - \tilde{\nu}(t_0) = \hat{\nu}(t_0) - \tilde{\nu}(t_0),$$

and by assumption this element belongs to $H^{2m-1}(X_0, \mathbb{Z})$.

Claim: If $\eta \in H^{2m-1}(X_0, \mathbb{C})$ and $\rho(\gamma)(\eta) - \eta \in H^{2m-1}(X_0, \mathbb{Z})$ for all $\gamma \in \pi_1(T, t_0)$, then $\eta \in H^{2m-1}(X_0, \mathbb{Q})$.

Note that the proof will be finished if we verify this Claim. To this end, we view X_0 as a hyperplane section of a smooth complete intersection $Y_0 \subset \mathbb{P}^{2m+r+1}$ of dimension $2m$ and multidegree $(d_0, \ldots, d_r)$. Set $L = \mathcal{O}_Y(1)$. The linear system $|L|$ corresponds to a projective linear subspace of $\mathbb{P}H^0(\mathbb{P}^n, E)$. Let $\Delta_L \subset |L|$ be the discriminant locus, and define $U_L = |L| \setminus \Delta_L$. It is known that $\Delta_L \subset |L|$ is an irreducible hypersurface (cf. [BeS, Lemma 1.6.5]). Choose a Lefschetz pencil $\mathbb{P}^1 \subset |L|$ of hyperplane sections of Y_0 that passes through the point $0 \in |L|$, and denote the discriminant locus in $\mathbb{P}H^0(\mathbb{P}^n, E)$ by Δ_E. As $\Delta_L = \Delta_E \cap |L|$, it follows that $\mathbb{P}^1 \cap \Delta_E = \mathbb{P}^1 \cap \Delta_L = \{t_1, \ldots, t_k\}$

is a finite set of points. The fundamental group of $U_L \cap \mathbb{P}^1 = \mathbb{P}^1 \setminus \{t_1, \ldots, t_k\}$ has standard generators γ_i winding once around t_i. Let $\delta_i \in H^{2m-1}(X_0, \mathbb{Z})$ be the vanishing cocycle associated to γ_i. Since $g_*\pi_1(T, t_0) \subset \pi_1(U, 0)$ has finite index, N say, we have $\gamma_i^N = g_*\tilde{\gamma}_i$ for $i = 1, \ldots, k$. According to the Picard–Lefschetz formula, the action of $\tilde{\gamma}_i$ via the monodromy representation is given by

$$\rho(\tilde{\gamma}_i)(\eta) = \eta \pm N\langle \eta, \delta_i \rangle \delta_i.$$

Hence we find that $\rho(\tilde{\gamma}_i)(\eta) - \eta = \pm N\langle \eta, \delta_i \rangle \delta_i \in H^{2m-1}(X_0, \mathbb{Z})$ for $i = 1, \ldots, k$. Thus $\langle \eta, \delta_i \rangle \in \mathbb{Q}$ for $i = 1, \ldots, k$. The pairing

$$\langle \, , \rangle : H^{2m-1}(X_0, \mathbb{Q}) \times H^{2m-1}(X_0, \mathbb{Q}) \to \mathbb{Q}$$

is non–degenerate over $\mathbb{Q}$, and induces an isomorphism

$$H^{2m-1}(X_0, \mathbb{Q}) \xrightarrow{\sim} \mathrm{Hom}(H^{2m-1}(X_0, \mathbb{Q}), \mathbb{Q}))$$

sending an element $\alpha \in H^{2m-1}(X_0, \mathbb{Q})$ to $\langle \alpha, - \rangle$. As the vanishing cocycles $\delta_1, \ldots, \delta_k$ generate $H^{2m-1}(X_0, \mathbb{Q})$ (see for instance [V4, Lecture 4, 2.3] or [DK, Exposé XVIII, 6.6.1]), it follows that $\langle \eta, \lambda \rangle \in \mathbb{Q}$ for all $\lambda \in H^{2m-1}(X_0, \mathbb{Q})$; hence $\eta \in H^{2m-1}(X_0, \mathbb{Q})$. $\square$

2.4 Main result

We formulate and prove the main result of this Chapter, which extends the aforementioned theorem of Green–Voisin to the case of complete intersections in projective space.

Theorem 2.4.1. *Let $X = V(d_0, \ldots, d_r) \subset \mathbb{P}^{2m+r}$ be a smooth complete intersection of odd dimension $2m-1$ ($m \geq 2$) and multidegree $(d_0, \ldots, d_r)$ ($d_0 \geq \ldots \geq d_r$, $d_i \geq 2$ for $i = 0, \ldots, r$). If X is very general, then the image of the Abel–Jacobi map*

$$\psi_X : \mathrm{CH}^m_{\mathrm{hom}}(X) \to J^m(X)$$

is contained in the torsion points of $J^m(X)$, unless we are in one of the following cases:

(i) $(r = 0)$ $X = V(d) \subset \mathbb{P}^4$ $(3 \leq d \leq 5)$, $X = V(3) \subset \mathbb{P}^6$, $X = V(3) \subset \mathbb{P}^8$.

(ii) $(r = 1)$ $X = V(3,3) \subset \mathbb{P}^5$.

(iii) $(r = 1)$ $X = V(d,2) \subset \mathbb{P}^{2m+1}$, $d \geq 2$, $m \geq 2$.

(iv) $(r = 2)$ $X = V(d,2,2) \subset \mathbb{P}^{2m+2}$, $d \geq 2$, $m \geq 2$.

(v) $(r = 3)$ $X = V(2,2,2,2) \subset \mathbb{P}^{2m+3}$, $m \geq 2$.

Proof: We have seen that if $X_0 = V(d_0, \ldots, d_r) \subset \mathbb{P}^{2m+r}$ is a very general complete intersection, every cycle $Z_0 \in Z^m_{\mathrm{hom}}(X_0)$ can be spread out to a relative cycle $Z_T \in Z^m_{\mathrm{hom}}(X_T/T)$ after taking a finite étale covering $T \to U$ of the parameter space. If the conditions (1) and (2) of Corollary 2.3.8 are satisfied, the normal function $\nu \in H^0(T, \mathcal{J}^m_h)$ associated to Z_T is torsion by Lemmas 2.3.9 and 2.3.10. Note that this is the case if

$$(m + r - 1)d_r \geq 2m + r + 2 = 2(m + r - 1) + 4 - r,$$

that is, if

$$d_r \geq 2 + \frac{4 - r}{m + r - 1}.$$

For $r = 0$ this condition is

$$d_0 \geq 2 + \frac{4}{m - 1}.$$

This is the result for hypersurfaces of odd degree in projective space obtained by Green and Voisin. The only exceptions are the ones listed in (i); see [G3].

Note that the Abel–Jacobi map is trivial for quadric hypersurfaces, since their intermediate Jacobians vanish.

For $r \geq 1$, $m \geq 2$ we have $\frac{4-r}{m+r-1} \leq 2$. Therefore we are done if $d_r \geq 4$, and it remains to check the cases $d_r = 2$ and $d_r = 3$.

Case 1. $d_r = 2$

(1) $d_0 + \ldots + d_{r-1} + 2m - 2 \geq 2m + r + 2$

(2) $d_1 + \ldots + d_{r-1} + 2m \geq 2m + r + 1$.

Since $d_i \geq 2$ for $i = 0, \ldots, r$, condition (1) is always satisfied if $r \geq 4$; condition (2) is always satisfied if $r \geq 3$. We check the cases $r = 1$, $r = 2$ and $r = 3$ separately:

* $r = 1$

If $(d_0, d_1) = (d, 2)$, then the condition (1) is satisfied if $d_0 \geq 5$, but (2) is never satisfied.

* $r = 2$

(1) $d_0 + d_1 \geq 6$

(2) $d_1 \geq 3$.

For $(d_0, d_1, d_2) = (d, 2, 2)$, $d \geq 2$, the condition (2) is never satisfied. If $d_1 \geq 3$, then (1) and (2) are satisfied.

* $r = 3$

(1) $d_0 + d_1 + d_2 \geq 7$

(2) $d_1 + d_2 \geq 4$.

We see that condition (2) is always satisfied; condition (1) is satisfied unless $(d_0, d_1, d_2, d_3) = (2, 2, 2, 2)$.

Case 2. $d_r = 3$

(1) $d_0 + \ldots + d_{r-1} + 3m - 3 \geq 2m + r + 2$

(2) $d_1 + \ldots + d_{r-1} + 3m \geq 2m + r + 1$.

As in this case $d_i \geq d_r = 3$ for $i = 0, \ldots, r$, condition (1) is satisfied if $m + 2r \geq 5$ and condition (2) is satisfied if $m + 2r \geq 4$. Hence (1) and (2) are satisfied if $m \geq 2$ and $r \geq 2$. The only remaining case is $m = 2$, $r = 1$:

(1) $d_0 + d_1 \geq 7$

(2) $2d_1 \geq 6$.

Both conditions are satisfied unless $(d_0, d_1) = (3, 3)$. $\square$

Remark 2.4.2. Let us consider the exceptional cases (i)–(v):

(i) The cubic and quartic threefold are Fano threefolds that contain a positive–dimensional family F of lines; in both cases, the Abel–Jacobi map $\mathrm{Alb}(F) \to J^2(X)$ is surjective (cf. [Ty1], [CG] and [BlM]). The cubic fivefold $X = V(3) \subset \mathbb{P}^6$ contains a family F of 2–planes; Collino [Col] showed that $\mathrm{Alb}(F) \xrightarrow{\sim} J^3(X)$. For a very general quintic threefold $X = V(5) \subset \mathbb{P}^4$, the image of the Abel–Jacobi map is non–torsion; see [Gr2] and [CC]. Clemens [C1] showed that the image of the Abel–Jacobi map is not even finitely generated; his proof is based on monodromy arguments. Voisin [V3] has given a different proof of this statement using infinitesimal methods. The image of the Abel–Jacobi map is also not finitely generated for a very general cubic sevenfold $X = V(3) \subset \mathbb{P}^8$; see [AC].

(ii) For a very general intersection of two cubics $X = V(3,3) \subset \mathbb{P}^5$, the image of ψ_X is not finitely generated; see [BaM], [Par].

(iii) This case is covered by the following result:

Theorem. *Let Y be a smooth projective variety of even dimension $2m$, and let L be a very ample line bundle on Y. Suppose that $X \in |L|$ is a general smooth divisor. If*

(i) $H^{2m-1}_{\mathrm{var}}(X) \neq 0$

(ii) $\mathrm{im}\,\mathrm{cl}_{Y,\mathbb{Q}} \cap H^{2m}_{\mathrm{pr}}(Y,\mathbb{Q}) \neq 0$

then $\mathrm{im}\,\psi_{X,\mathbb{Q}} \neq 0$.

This result is essentially due to Griffiths, N. Katz and Zucker; see [DK, Exposé XVIII, Cor. 5.8.7]. Let $U \subset |L|$ be the smooth part, and let $V_{\mathbb{Q}}$ be the local system of variable cohomology. Let $\mathbb{P}^1 \subset |L|$ be a Lefschetz pencil, with smooth part $U_0 = U \cap \mathbb{P}^1$. Katz shows that there is an injective map

$$H^{2m}_{\mathrm{pr}}(Y,\mathbb{Q}) \to H^1(U_0, V_{\mathbb{Q}}).$$

As this map sends the class $[Z]$ to the cohomological invariant $\partial\nu_Z$ of the associated normal function ν_Z (see [Z1, Prop. 3.9]), the desired statement follows. Note that we can replace (ii) by $\mathrm{Hdg}^m_{\mathrm{pr}}(Y)_{\mathbb{Q}} \neq 0$, as it is possible to associate a normal function to a primitive Hodge class on Y (cf. [G4, Lecture 6]). In a similar way one can deduce the non–vanishing of the infinitesimal invariant $\delta\nu_Z$; see [MuS].

If Y is a quadric of dimension $2m$ and $X = Y \cap V(d)$ is a smooth hypersurface section, the conditions of the previous theorem are satisfied if $Z = Z_1 - Z_2$ is the difference of two m–planes that belong to the different rulings of Y (note that X has nontrivial vanishing cohomology; see [DK, Exposé XI]).

(iv) This case can be handled in the same way as (ii). If $Y = V(2,2) \subset \mathbb{P}^{2m+2}$ is a complete intersection of two quadrics, it is known that Y contains exactly 4^{m+1} m–planes; the cohomology classes of the differences of these m–planes generate $H^{2m}_{\mathrm{pr}}(Y, \mathbb{Q})$ [Re].

(v) For $m = 2$, it is known that the image of ψ_X is non–torsion if X is very general. This follows from a result of Bardelli [Bar]. Using a generalization of Bardelli's techniques, we shall show in Chapter 3 that the image of ψ_X is non–torsion for every $m \geq 2$.

Remark 2.4.3.

(1) The cases (iii) and (iv) mentioned above are the only cases where the technique of Katz produces non–torsion normal functions, in view of the cohomological Noether–Lefschetz theorem (see [DK, Exposé XIX]). For Calabi–Yau complete intersections and cubic sevenfolds (which can in some sense be interpreted as the 'mirrors' of rigid Calabi–Yau threefolds [AC]) one uses a similar technique, based on Mark Green's Lemma (see [Kim] or [V4, Lecture 3]), which produces a countable union of 'good' components of the Noether–Lefschetz locus whose union is dense in the parameter space. For details, see [V3] or [BaM].

(2) Let $V \subset U = \mathbb{P}H^0(\mathbb{P}^{2m}, \mathcal{O}_{\mathbb{P}}(d)) \setminus \Delta$ be a Zariski open subset of the complement of the discriminant locus for the family of hypersurfaces of degree d in $\mathbb{P}^{2m}$. If $d \geq 2$, then there are no nonzero normal functions that are defined over V. This is clear if $d = 2$; for $d \geq 3$ it is proved in [GH2, §3], using a result of N. Katz on cohomology with values in the local system of vanishing cohomology over a Lefschetz pencil (see [DK, Exposé XVIII, Th. 5.7]) and results of Zucker on normal functions defined over Lefschetz pencils (see [Z1, Thm. (4.17) and Cor. (4.52)]). A similar argument applies in the case of complete intersections $X = V(d_0, \ldots, d_r)$ such that $d_i \geq 2$ for all $i = 0, \ldots, r$.

Chapter 3

Complete intersections of four quadrics

3.1 Introduction

Intersections of two or three quadrics in projective space have been studied extensively (cf. [PB], [Re], [Te1] and [Ty2]). Suppose that X is a complete intersection of two or three quadrics and $\dim X = 2m-1$ ($m \geq 2$). Then $H^{2m-1}(X)$ carries a Hodge structure of level one, and the intermediate Jacobian $J^m(X)$ is parametrized by a family of codimension m cycles, as predicted by the generalized Hodge conjecture. If $X = V(2,2,2,2) \subset \mathbb{P}^{2m+3}$ ($m \geq 2$) is a complete intersection of four quadrics, then $H^{2m-1}(X)$ carries a Hodge structure of level three; hence the Abel–Jacobi map

$$\psi_X : \mathrm{CH}^m_{\mathrm{hom}}(X) \longrightarrow J^m(X)$$

cannot be surjective. For very general X, the algebraic part $J^m_{\mathrm{alg}}(X)$ of the intermediate Jacobian is zero (cf. Theorem 0.0.5), and the image of the Abel–Jacobi map is (at most) a countable subgroup of $J^m(X)$.

If $m = 2$, $X \subset \mathbb{P}^7$ is a Calabi–Yau threefold. Bardelli studied a family $\{X_t\}$ of this type with X_t invariant under a natural involution $\sigma : \mathbb{P}^7 \to \mathbb{P}^7$. In this case, the anti–invariant part $H^3(X_t)^-$ is a sub Hodge structure of level one in $H^3(X_t)$. Bardelli showed that the corresponding abelian variety $J^2(X_t)^-$ (a subtorus of $J^2(X_t)$) is parametrized by a family of codimension m cycles on X_t; see [Bar]. By a specialization argument he concluded that $\mathrm{im}\,\psi_X$ is non–torsion for very general X. Voisin [V6] proved a similar statement for arbitrary Calabi–Yau threefolds.

We show that the image of ψ_X is not contained in the torsion points of $J^m(X)$ if $X = V(2,2,2,2) \subset \mathbb{P}^{2m+3}$ ($m \geq 2$) is very general. As a consequence, the degree bounds obtained in Theorem 2.4.1 are sharp. The results in this chapter are based on a slight modification of Bardelli's techniques; therefore we have chosen to preserve his notation, whenever possible. To obtain nontrivial cycles on X, the idea is to consider singular quadrics that contain X. If X is general, the web of quadrics that corresponds to X will contain a finite number of quadrics of corank 2, i.e., of rank $2m+2$. A quadric of rank $2m+2$ is a cone with one–dimensional vertex over a smooth quadric in $\mathbb{P}^{2m+1}$, hence it contains two rulings of projective $(m+2)$–planes. We obtain interesting codimension m cycles on X by intersecting the difference of two $(m+2)$–planes from the different rulings with the three remaining quadrics that define X. By construction these cycles are homologically trivial. To show that their image under the Abel–Jacobi map is nonzero, we define a natural involution σ on $\mathbb{P}^{2m+3}$ and specialize to complete intersections of σ–invariant quadrics. As general complete intersections of four σ–invariant quadrics contain a one–dimensional family of interesting cycles, we can show the nontriviality of the Abel–Jacobi mapping for these varieties using infinitesimal methods.

In Section 3.2 we study the family of cycles on general complete intersections of four σ–invariant quadrics. Section 3.3 is devoted to the infinitesimal Abel–Jacobi mapping and the specialization argument. The results in this chapter have previously appeared as report W96–09 at the University of Leiden.

3.2 Construction of a family of cycles

Let $X = V(2,2,2,2) \subset \mathbb{P}^{2m+3}$ $(m \geq 2)$ be an odd–dimensional smooth complete intersection of four quadrics. In this section we consider quadrics that are invariant under the involution

$$\sigma : \mathbb{P}^{2m+3} \to \mathbb{P}^{2m+3}, \quad \sigma(x,y) = (x,-y)$$

where $(x,y) = (x_0 : \ldots : x_{m+1} : y_0 : \ldots : y_{m+1})$ are homogeneous coordinates on $\mathbb{P}^{2m+3}$. The fixed point locus of σ is

$$\mathrm{Fix}(\sigma) = \mathbb{P}^{m+1}_x \amalg \mathbb{P}^{m+1}_y,$$

where $\mathbb{P}^{m+1}_x = \{(x,y) \in \mathbb{P}^{2m+3} : y = 0\}$ and $\mathbb{P}^{m+1}_y = \{(x,y) \in \mathbb{P}^{2m+3} : x = 0\}$. If X is general, then $\dim\left(X \cap \mathrm{Fix}(\sigma)\right) = m-3$. Set

$$\tilde{V} \;\; = \;\; H^0(\mathbb{P}^{2m+3}, \mathcal{O}_{\mathbb{P}}(2))$$

and let $\tilde{V} \subset V$ be the linear subspace of σ–invariant quadrics. Note that

$$\dim \tilde{V} = \binom{2m+5}{2}, \ \dim V = (m+2)(m+3).$$

To produce cycles on X, we consider the varieties

$$\mathcal{L}_k(X) = \{Q \in \mathbb{P}(\tilde{V}) : V(Q) \supset X \text{ and } \operatorname{rank} Q \leq k\}$$

of singular quadrics containing X. Note that $\mathcal{L}_{2m}(X) = \emptyset$ since X is smooth. If $Q \in V$ then $Q(x,y) = Q'(x) + Q''(y)$, i.e., the corresponding symmetric matrix Q is of the form

$$Q = \begin{pmatrix} Q' & 0 \\ 0 & Q'' \end{pmatrix}.$$

Suppose that $X = \bigcap_{i=0}^{3} V(Q_i)$, $Q_i \in V$, is a complete intersection of four σ–invariant quadrics. Let $M = \langle Q_0, \ldots, Q_3 \rangle$ be the web spanned by $Q_0, \ldots, Q_3$, and let $\lambda = (\lambda_0, \ldots, \lambda_3)$ be homogeneous coordinates on $M \cong \mathbb{P}^3$. A quadric $Q(\lambda) \in M$ is of the form

$$Q(\lambda) = \sum_{i=0}^{3} \lambda_i Q_i, \quad Q_i(x,y) = Q_i'(x) + Q_i''(y).$$

Consider the following subvarieties of $\mathbb{P}^3$:

$$\begin{aligned} S' &= \{\lambda \in \mathbb{P}^3 : \operatorname{rank}(\sum_i \lambda_i Q_i') \leq m\} \\ S'' &= \{\lambda \in \mathbb{P}^3 : \operatorname{rank}(\sum_i \lambda_i Q_i'') \leq m\} \\ K' &= \{\lambda \in \mathbb{P}^3 : \det(\sum_i \lambda_i Q_i') = 0\} \\ K'' &= \{\lambda \in \mathbb{P}^3 : \det(\sum_i \lambda_i Q_i'') = 0\}. \end{aligned}$$

Set $C = K' \cap K''$. It is clear that

$$\mathcal{L}_{2m+2}(X) = S' \cup S'' \cup C.$$

Set $V_x = H^0(\mathbb{P}^{m+1}_x, \mathcal{O}_{\mathbb{P}}(2))$, $V_y = H^0(\mathbb{P}^{m+1}_y, \mathcal{O}_{\mathbb{P}}(2))$. The rational maps

$$p' : \mathbb{P}(V) - - \to \mathbb{P}(V_x), \quad p'(Q) = Q'$$

and

$$p'' : \mathbb{P}(V) - - \to \mathbb{P}(V_y), \quad p''(Q) = Q''$$

induce a rational map

$$p' \times p'' : \mathbb{P}(V) - - \to \mathbb{P}(V_x) \times \mathbb{P}(V_y) \cong \mathbb{P}^{\binom{m+3}{2}-1} \times \mathbb{P}^{\binom{m+3}{2}-1}$$

with one–dimensional fibers. Let $\Delta_{x,1} \subset \mathbb{P}(V_x)$ be the discriminant locus. It is stratified by irreducible subvarieties

$$\Delta_{x,i} = \{Q' \in \mathbb{P}(V_x) : \text{corank } Q \geq i\} = \{Q' \in \mathbb{P}(V_x) : \dim(\text{Sing } Q') \geq i-1\}$$

of codimension $\binom{i+1}{2}$ in $\mathbb{P}(V_x)$. There is a similar stratification of $\Delta_{y,1} \subset \mathbb{P}(V_y)$ by irreducible subvarieties $\Delta_{y,i}$. The discriminant locus $\Delta_1 \subset \mathbb{P}(V)$ consists of two irreducible components $\Delta_{1,0}$ and $\Delta_{0,1}$ of codimension one:

$$\begin{aligned} \Delta_{1,0} &= \{(Q', Q'') \in \mathbb{P}(V) : \det Q' = 0\} \\ \Delta_{0,1} &= \{(Q', Q'') \in \mathbb{P}(V) : \det Q'' = 0\}. \end{aligned}$$

We can write $\Delta_2 = \Delta_{2,0} \cup \Delta_{1,1} \cup \Delta_{0,2}$, where

$$\begin{aligned} \Delta_{2,0} &= \{(Q', Q'') \in \mathbb{P}(V) : \text{rank } Q' \leq m\} \\ \Delta_{1,1} &= \{(Q', Q'') \in \mathbb{P}(V) : \det Q' = \det Q'' = 0\} \\ \Delta_{0,2} &= \{(Q', Q'') \in \mathbb{P}(V) : \text{rank } Q'' \leq m\}. \end{aligned}$$

In general, the subvarieties $\Delta_k = \{Q \in \mathbb{P}(V) : \dim \text{Sing}(Q) \geq k-1\}$ admit a similar decomposition

$$\Delta_k = \Delta_{k,0} \cup \ldots \cup \Delta_{0,k}.$$

Lemma 3.2.1. $\Delta_{1,1}$ *is irreducible.*

Proof: Consider the correspondence

$$\tilde{\Delta}_{1,1} = \{(Q, p', p'') \in \mathbb{P}(V) \times \mathbb{P}_x^{m+1} \times \mathbb{P}_y^{m+1} : p' \in \text{Sing}(Q'), p'' \in \text{Sing}(Q'')\}$$

with projections $p_1 : \tilde{\Delta}_{1,1} \to \Delta_{1,1}$ and $p_2 : \tilde{\Delta}_{1,1} \to \mathbb{P}_x^{m+1} \times \mathbb{P}_y^{m+1}$. One readily verifies that $\tilde{\Delta}_{1,1} = \mathbb{P}(\pi_1^* S^2 \Omega^1_{\mathbb{P}_x} \oplus \pi_2^* S^2 \Omega^1_{\mathbb{P}_y})$ is a locally trivial fiber bundle over $\mathbb{P}_x^{m+1} \times \mathbb{P}_y^{m+1}$, where π_1 and π_2 are the projections of $\mathbb{P}_x^{m+1} \times \mathbb{P}_y^{m+1}$ onto the first and second factor. Hence $\tilde{\Delta}_{1,1}$ is smooth, connected and irreducible and its image $\Delta_{1,1} = p_1(\tilde{\Delta}_{1,1})$ is irreducible. $\square$

The irreducibility of the components $\Delta_{i,j}$ of Δ_k, $k \geq 2$, is proved in a similar way. Using the projection map $p' \times p''$, we compute $\operatorname{codim} \Delta_{i,j} = \operatorname{codim} \Delta_{x,i} + \operatorname{codim} \Delta_{y,j}$. Hence

$$\operatorname{codim} \Delta_{1,1} = 2, \quad \operatorname{codim} \Delta_{2,0} = \operatorname{codim} \Delta_{0,2} = 3$$
$$\operatorname{codim} \Delta_{2,1} = \operatorname{codim} \Delta_{1,2} = 4, \quad \operatorname{codim} \Delta_{3,0} = \operatorname{codim} \Delta_{0,3} = 6.$$

Note that $S' = \Delta_{2,0} \cap M$, $S'' = \Delta_{0,2} \cap M$, $K' = \Delta_{1,0} \cap M$, $K'' = \Delta_{0,1} \cap M$ and $C = \Delta_{1,1} \cap M$.

Let $T \subset G(4, \tilde{V})$ be the subset parametrizing smooth complete intersections of four quadrics in $\mathbb{P}^{2m+3}$, and let $U \subset G(4, V)$ be the subset parametrizing smooth complete intersections of four $\sigma-$ invariant quadrics in $\mathbb{P}^{2m+3}$. Let $X_U \to U$ be the universal family; we denote the fiber over $t \in U$ by X_t.

Lemma 3.2.2. *There exists a Zariski open subset $U^0 \subset U$ such that if $t \in U^0$, then $X = X_t$ satisfies*

(i) $\mathcal{L}_{2m+1}(X) = \emptyset$

(ii) *S' and S'' are finite sets of $\binom{m+3}{3}$ points.*

(iii) *S', S'' and C are mutually disjoint.*

Proof:

(i) Since $\operatorname{codim}(\Delta_3) = 4$, a general web $M \subset \mathbb{P}(V)$ does not intersect Δ_3.

(ii) The locus $\Delta_{x,2}$ has codimension 3 in $\mathbb{P}(V_x)$. Hence, if the web M is genral, the web $p'(M)$ will intersect $\Delta_{x,2}$ in a finite number of points. The degree of $\Delta_{x,2}$ is calculated in [HT, p. 81]. A similar argument applies to $\Delta_{y,2}$.

(iii) follows directly from (i).

□

Remark 3.2.3. Let N be a web of quadrics in $\mathbb{P}^n$, and let $\Delta \subset N$ be its discriminant locus. Let us compute the tangent space to Δ at Q, where $Q \in \Delta$ is a rank n quadric. We may suppose that $N = \langle Q, Q_1, Q_2, Q_3 \rangle$, $Q_i = (q^i_{ab})_{a,b}$, and that Q has the form

$$Q = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & \\ \vdots & & \ddots & \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Let (t_1, t_2, t_3) be local coordinates on N around Q. The local equation of Δ at Q is

$$\Delta = \{(t_1, t_2, t_3) : \det(Q + \sum_{i=1}^{3} t_i Q_i) = 0\},$$

and its linear term defines the tangent space $T_Q\Delta$; hence

$$T_Q\Delta = \{(t_1, t_2, t_3) : \sum_{i=1}^{3} t_i q_{00}^i = 0\}$$

is the vector space of quadrics in N containing the singular point $P = (1, 0, \ldots, 0)$ of Q. This shows that

$$Q \in \Delta \text{ is a smooth point} \iff P \text{ is not a base point of } N.$$

Lemma 3.2.4. *$K' \setminus S'$ and $K'' \setminus S''$ are smooth.*

Proof: Consider the web $p'(M)$ on $\mathbb{P}_x^{m+1}$. Suppose that $Q' \in K' \setminus S'$, i.e., rank $Q' = m + 1$. By Remark 3.2.3, we have to show that $\mathrm{Sing}(Q') = \{P_0'\}$ is not a base point of $p'(M)$. As the quadrics in $p'(M)$ are obtained by restricting the quadrics in M to $\mathbb{P}_x^{m+1}$, we have $\mathrm{Bs}(p'(M)) = \mathrm{Bs}(M) \cap \mathbb{P}_x^{m+1} = X \cap \mathbb{P}_x^{m+1}$. One easily checks that $\mathrm{Sing}(Q') = \mathrm{Sing}(Q) \cap \mathbb{P}_x^{m+1}$; hence $P_0' \notin \mathrm{Bs}(p'(M))$ because $\mathrm{Sing}(Q) \cap X = \emptyset$. The same argument works for $K'' \setminus S''$. $\square$

Lemma 3.2.5. *If $X = X_t$ with $t \in U^0$, then*

(i) C is connected.

(ii) T_QK' and T_QK'' intersect transversally at every point $Q \in C$.

So C is a smooth, irreducible curve.

Proof: (i) follows from the Lefschetz hyperplane theorem. For (ii) we note that if $Q \in C$, then rank $Q' =$ rank $Q'' = m + 1$; hence K' and K'' are smooth at Q by Lemma 3.2.5. We may assume that

$$Q' = Q'' = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix}.$$

If $T_Q K'$ and $T_Q K''$ do not intersect transversally they coincide. This means that there exists some $\lambda \in \mathbb{C}^*$ such that $(q^i_{00})'' = \lambda (q^i_{00})'$ for $i = 1, 2, 3$. It follows that the variety X_t defined by the equations

$$\sum_{i=1}^{m+1} x_i^2 + \sum_{i=1}^{m+1} y_i^2 = 0$$
$$\sum_{a,b=0}^{m+1} (q^i_{ab})' x_a x_b + \sum_{a,b=0}^{m+1} (q^i_{ab})'' y_a y_b = 0 \qquad i = 1, 2, 3$$

has a singular point at $(1, 0, \ldots, 0, \sqrt{\frac{-1}{\lambda}}, 0, \ldots, 0)$, contradiction. $\square$

Remark 3.2.6. The previous lemma shows that if $t \in U^0$, $C = C_t$ is a smooth complete intersection of two surfaces of degree $m+2$ in $\mathbb{P}^3$; the genus of C is $m^3 + 4m^2 + 4m + 1$.

We define

$$\mathcal{L}^{m+2}_{2m+2}(X) = \{(Q, \Lambda) \in \mathcal{L}_{2m+2}(X) \times G(m+3, 2m+4) : \Lambda \subset V(Q)\}.$$

The fiber over Q of the natural projection map

$$f : \mathcal{L}^{m+2}_{2m+2}(X) \to \mathcal{L}_{2m+2}(X) = S' \cup S'' \cup C$$

is the Fano variety $F_{m+2}(Q)$ of $(m+2)$–planes contained in Q; it is a smooth variety of dimension $\binom{m+1}{2}$ that has one connected component if rank $Q = 2m+1$ and two connected components if rank $Q = 2m+2$. The components of $F_{m+2}(Q)$ are rational, since they can be covered by open sets U_α that are isomorphic to the affine space of skew–symmetric $(m+1) \times (m+1)$–matrices. Consider the Stein factorization

$$\mathcal{L}_C(X) \xrightarrow{g} \tilde{C} \xrightarrow{h} C$$

of the restriction of $f : \mathcal{L}^{m+2}_{2m+2}(X) \to \mathcal{L}_{2m+2}(X)$ to $\mathcal{L}_C(X) = f^{-1}(C)$. If $X = X_t$, $t \in U^0$, then $h : \tilde{C}_t \to C_t$ is an unramified double covering and $\tilde{C}_t$ is smooth. There is a natural involution σ' on $\tilde{C}$ that interchanges the two rulings of $(m+2)$–planes.

Remark 3.2.7. The involution σ' on $\tilde{C}$ need not be induced by the involution σ on $\mathbb{P}^{2m+3}$. To see this, note that a quadric $Q \subset \mathbb{P}^{2m+3}$ of rank $2m+2$ is a cone over a smooth quadric $\hat{Q} \subset \mathbb{P}^{2m+1}$. By a suitable choice of coordinates we may assume that this $\mathbb{P}^{2m+1}$ is given by $x_0 = y_0 = 0$ and that $\hat{Q}$ is defined by

$$\sum_{i=1}^{m+1} x_i^2 + \sum_{i=1}^{m+1} y_i^2 = 0.$$

Consider the m–planes

$$\Lambda_1 = V(x_1 + \sqrt{-1}y_1, \dots, x_{m+1} + \sqrt{-1}y_{m+1})$$

and

$$\Lambda_2 = V(x_1 - \sqrt{-1}y_1, \dots, x_{m+1} - \sqrt{-1}y_{m+1}).$$

These two m–planes Λ_1 and Λ_2 are contained in $\hat{Q}$ and belong to the same ruling if and only if $\dim(\Lambda_1 \cap \Lambda_2) \equiv m (\operatorname{mod} 2)$. Since Λ_1 and Λ_2 do not intersect, they belong to different rulings if and only if m is even (e.g. in Bardelli's case $m = 2$). Taking the span of Λ_1 (resp. Λ_2) with the one–dimensional vertex of Q we obtain an $(m+2)$–plane Λ_1' (resp. Λ_2') in $\mathbb{P}^{2m+3}$ given by the same equations. The planes Λ_1' and Λ_2' are permuted by the involution σ, but they belong to the same ruling of Q if m is odd.

The idea of the proof of the next result is taken from a paper of Tyurin [Ty2]. A linear subspace of maximal dimension contained in a quadric is called a generator.

Lemma 3.2.8. *If $t \in U^0$ is general, then $\tilde{C}_t$ is irreducible.*

Proof: Let $\tilde{\Delta}_{1,1}$ be the smooth variety introduced in the proof of Lemma 3.2.1. Consider the correspondence

$$I = \{((Q, p', p''), L) \in \tilde{\Delta}_{1,1} \times G(m+1, 2m+4) : \langle p', p'', L \rangle \subset V(Q)\}$$

with projections $\pi_1 : I \to \tilde{\Delta}_{1,1}$ and $\pi_2 : I \to G(m+1, 2m+4)$. Let

$$I \longrightarrow \mathrm{Gen} \xrightarrow{p} \tilde{\Delta}_{1,1}$$

be the Stein factorization of π_1. The double covering $p : \mathrm{Gen} \to \tilde{\Delta}_{1,1}$ corresponds to the choice of a system of generators on a quadric Q of corank at least two; it is ramified over $\tilde{\Delta}_{1,1} \cap p_1^{-1}(\Delta_3)$. As the projection $p_1 : \tilde{\Delta}_{1,1} \to \Delta_{1,1}$ induces an isomorphism

$$\tilde{\Delta}_{1,1} \setminus (\tilde{\Delta}_{1,1} \cap p_1^{-1}(\Delta_3)) \xrightarrow{\sim} \Delta_{1,1} \setminus \ (\Delta_{1,1} \cap \Delta_3),$$

there is a unique lifting $C_t \subset \tilde{\Delta}_{1,1}$ of C_t if $t \in U^0$. I do not know whether Gen is irreducible, but for our purposes it suffices to exhibit an irreducible surface S_{Gen} containing $\tilde{C}_t$. Let $K \supset M$ be a 4–plane in $\mathbb{P}(V)$ such that $W = \mathrm{Bs}(K)$ is smooth and such that K intersects the loci $\Delta_{2,1}$ and $\Delta_{1,2}$ in a finite number of points but does not meet Δ_4. Set $Y' = K \cap \Delta_{1,0}$ and $Y'' = K \cap \Delta_{0,1}$. The surface

$$S = K \cap \Delta_{1,1} = Y' \cap Y''$$

has a finite number of ordinary double points. Set $\tilde{S} = p_1^{-1}(S) \subset \tilde{\Delta}_{1,1}$.

Claim. $p_1 : \tilde{S} \to S$ *is the desingularization of* S.

To prove the Claim, we consider a singular point $Q_s \in S$ with $Q_s \in \Delta_{2,1} \setminus (\Delta_{2,1} \cap \Delta_4)$. In this case $\mathrm{Sing}\, Q'_s = \ell'$ is a line and $\mathrm{Sing}\, Q''_s = \{p''\}$ is a point. To show that $\tilde{S}$ is obtained by blowing up the double points of S we have to identify the fiber $p_1^{-1}(Q_s) \cong \ell'$ with the projectivized tangent cone $\mathbb{P}TC_{Q_s}S$. We may assume that

$$Q'_s = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & & \dots & 1 \end{pmatrix}, \; Q''_s = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & 1 \end{pmatrix}.$$

Consider the universal family

$$Q(W) = \{(\lambda, x) \in K \times \mathbb{P}^{2m+3} : x \in Q_\lambda\}$$

of quadrics in K, defined by the equation $\lambda_0 Q_0(x) + \ldots + \lambda_4 Q_4(x) = 0$. One readily verifies that $Q(W)$ is smooth since W is smooth. Let $t_i = \frac{\lambda_i}{\lambda_0}$ $(i = 1, \ldots, 4)$ be affine coordinates on K around $Q_s = Q_0$. The quadric $Q_t = Q_0 + t_1 Q_1 + \ldots + t_4 Q_4$ decomposes as $Q_t = Q'_t + Q''_t$, where

$$Q'_t = \begin{pmatrix} \sum_{k=1}^4 (q^k_{00})' t_k & \sum_{k=1}^4 (q^k_{01})' t_k & \dots & \dots \\ \sum_{k=1}^4 (q^k_{01})' t_k & \sum_{k=1}^4 (q^k_{11})' t_k & \dots & \dots \\ \dots & \dots & 1 + \sum_{k=1}^4 (q^k_{22})' t_k & \dots \\ \vdots & & & \ddots \end{pmatrix}$$

and

$$Q''_t = \begin{pmatrix} \sum_{k=1}^4 (q^k_{00})'' t_k & \dots & \dots & \dots \\ \dots & 1 + \sum_{k=1}^4 (q^k_{11})'' t_k & \dots & \dots \\ \vdots & & \ddots & \end{pmatrix}.$$

Since $W = \mathrm{Bs}(K)$ is smooth, the system of equations

$$\begin{aligned}(q^1_{00})'x_0^2 + 2(q^1_{01})'x_0x_1 \quad &+ (q^1_{11})'x_1^2 + (q^1_{00})''y_0^2 = 0\\ &\vdots\\ (q^4_{00})'x_0^2 + 2(q^4_{01})'x_0x_1 \quad &+ (q^4_{11})'x_1^2 + (q^4_{00})''y_0^2 = 0\end{aligned}$$

has no non–trivial solutions; hence we may assume that

$$\begin{pmatrix} (q^1_{00})' & (q^1_{01})' & (q^1_{11})' & (q^1_{00})'' \\ \vdots & \vdots & \vdots & \vdots \\ (q^4_{00})' & (q^4_{01})' & (q^4_{11})' & (q^4_{00})'' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The tangent cone $TC_{Q_s}Y'$ is defined by the equation $t_1t_3 - t_2^2 = 0$; the tangent space $T_{Q_s}Y''$ is defined by $t_4 = 0$. Hence

$$TC_{Q_s}S = TC_{Q_s}Y' \cap T_{Q_s}Y''$$

is the cone in $T_{Q_s}Y'' \cong \mathbb{C}^3$ given by $t_1t_3 - t_2^2 = 0$. The restriction of $Q_t \in K$ to ℓ' is the quadric defined by

$$t_1x_0^2 + 2t_2x_0x_1 + t_3x_1^2 = 0.$$

In this way the net of quadrics on ℓ' can be identified with the projective space $\mathbb{P}^2$ with coordinates (t_1, t_2, t_3) and the projectivized tangent cone $\mathbb{P}TC_{Q_s}S$ is identified with the plane conic of degenerate quadrics on ℓ'. The identification of ℓ' with $\mathbb{P}TC_{Q_s}S$ is then given by the Veronese embedding of ℓ' in $\mathbb{P}^2$. This proves the Claim.

Set $S_{\mathrm{Gen}} = p^{-1}(\tilde{S}) \subset \mathrm{Gen}$. Since the double covering $S_{\mathrm{Gen}} \to \tilde{S}$ is ramified along a smooth divisor D (the union of the exceptional divisors) S_{Gen} is smooth and connected, hence irreducible. If we compose the maps p and p_1 with the inclusion of S in K, we obtain a morphism $f : S_{\mathrm{Gen}} \to K$ such that $\tilde{C}_t = f^{-1}(M_t)$. Bertini's theorem shows that $\tilde{C}_t$ is irreducible if the web M_t is general; see [FL, Thm. 1.1]. $\square$

Remark 3.2.9. One can also try to show the irreducibility of $\tilde{C}_t$ using a degeneration argument, as in [Bar]. The idea is to construct a one–parameter family $\{M_t\}$ of webs in $\mathbb{P}(V)$ such that $M_t \cap \Delta_3 = \emptyset$ for $t \neq 0$, and such that $M_0 \cap \Delta_3 = M_0 \cap (\Delta_{2,1} \setminus \Delta_4)$ consists of one point. Let $\{C_t\}$ be the corresponding family of curves. The curves C_t are smooth for $t \neq 0$, and C_0 has one ordinary double point Q_0. Let $\{\tilde{C}_t\}$ be the associated family of double coverings. Since $h : \tilde{C}_t \to C_t$ is unramified for $t \neq 0$, $\tilde{C}_t$ is smooth.

The double covering $h : \tilde{C}_0 \to C_0$ is ramified over Q_0, and $h^{-1}(Q_0)$ consists of one point $\tilde{Q}_0$. If one can show that $\mathcal{O}_{\tilde{C}_0, Q_0}$ contains no nilpotent elements then $h^0(\mathcal{O}_{\tilde{C}_0}) = 1$, since $\tilde{C}_0$ is connected. By semicontinuity and flatness it then follows that $h^0(\mathcal{O}_{\tilde{C}_t}) = 1$ for general t, and hence for all t. Thus it would follow that $\tilde{C}_t$ is irreducible for $t \neq 0$.

A point $P = (Q, \Lambda) \in \mathcal{L}^{m+2}_{2m+2}(X)$ defines an algebraic cycle $Z_P = \Lambda \cap X$ on X.

Lemma 3.2.10. *If $t \in U$ and $P = (Q, \Lambda) \in \mathcal{L}_C(X_t)$ are general points, then* $\dim Z_P = m - 1$. *Moreover, Z_P is smooth.*

Proof: In the product

$$G(m+3, 2m+4) \times \mathbb{P}(V) \times G(3, V) \times G(4, V)$$

we have a correspondence

$$I = \{(\Lambda, Q, \pi, M) : \Lambda \subset V(Q), Q \in M \setminus \pi\}.$$

Set

$$T = \{(\Lambda, Q, \pi, M) \in I : \Lambda \text{ intersects } \operatorname{Bs}(\pi) \text{ non–transversally}\}.$$

By Bertini's theorem, T is a Zariski closed subset of I. Choose three quadrics Q_1, Q_2 and Q_3 in V such that $Y = \bigcap_{i=1}^{3} V(Q_i)$ is smooth, and choose an $(m+2)$–plane $W \subset \mathbb{P}^{2m+3}$ that meets Y, $\mathbb{P}^{m+1}_x$ and $\mathbb{P}^{m+1}_y$ transversely. Since

$$h^0(\mathbb{P}^{2m+3}, \mathcal{I}_W(2)) = \binom{2m+5}{2} - \binom{m+4}{2},$$

the two subspaces $\mathbb{P}(V)$ and $\mathbb{P}H^0(\mathbb{P}^{2m+3}, \mathcal{I}_W(2))$ of $\mathbb{P}(\tilde{V})$ intersect in a linear subspace of dimension at least d, where

$$d \geq (m+2)(m+3) - \binom{m+4}{2} - 1 = \frac{m(m+3)}{2} - 1.$$

Hence $\dim(\mathbb{P}(V) \cap \mathbb{P}H^0(\mathbb{P}^{2m+3}, \mathcal{I}_W(2))) \geq 4$ if $m \geq 2$, and we can choose a σ–invariant quadric Q_0 that satisfies the following conditions: Q_0 is not contained in $\pi_0 = \langle Q_1, Q_2, Q_3 \rangle$, $V(Q_0) \supset W$ and $X_0 = Y \cap V(Q_0)$ is smooth. By construction $Z = W \cap X_0 = W \cap Y$ is smooth of dimension $m - 1$. Set $M_0 = \langle Q_0, \ldots, Q_4 \rangle$. Since $(W, Q_0, \pi_0, M_0) \in I \setminus T$, the assertions of the Lemma follow. $\square$

Let $P(x)$ be the Hilbert polynomial of a smooth complete intersection of $m+1$ hyperplanes and three quadrics in $\mathbb{P}^{2m+3}$. Lemma 3.2.10 shows that if $X = X_t$, $t \in U$ general, there exist a Zariski open subset $\mathcal{L}_C(X)^0 \subset \mathcal{L}_C(X)$ and a well–defined map $i : \mathcal{L}_C(X)^0 \to \mathrm{Hilb}_X^{P(x)}$ to the Hilbert scheme of subschemes of X with Hilbert polynomial $P(x)$.

Lemma 3.2.11. *If $X = X_t$, $t \in U^0$ general, then the map $i : \mathcal{L}_C(X)^0 \to \mathrm{Hilb}_X^{P(x)}$ is injective.*

Proof: If $Z = \Lambda \cap X \in i(\mathcal{L}_C(X)^0)$, then Z is a complete intersection of three quadrics in Λ and Λ is spanned by Z. As the intersection

$$M \cap \mathbb{P}H^0(\mathbb{P}^{2m+3}, \mathcal{I}_\Lambda(2)) \subset \mathcal{L}_{2m+2}(X)$$

is a nonempty linear subspace of $\mathbb{P}(V)$ and $\mathcal{L}_{2m+2}(X) = S' \cup S'' \cup C$ contains no linear subspaces of dimension one, the quadric Q containing Λ is also uniquely determined. $\square$

Lemma 3.2.12. *If $X = X_t$, $t \in U$, then X is contained in a smooth complete intersection $Y = \bigcap_{i=1}^3 V(\tilde{Q}_i)$ of three σ–invariant quadrics.*

Proof: Consider the map

$$\nu : \mathbb{P}^{2m+3} \to \mathbb{P}(V^\vee), \quad \nu(x,y) = \{Q \in \mathbb{P}(V) : Q(x,y) = 0\}$$

and denote its image by $W \subset \mathbb{P}(V^\vee)$. Note that $W \cong \mathbb{P}^{2m+3}/ < \sigma >$ and $\mathrm{Sing}(W) \cong \nu(\mathbb{P}_x^{m+1} \amalg \mathbb{P}_y^{m+1})$. One has $\nu(X) = W \cap L$, $L = \mathrm{Ann}\langle Q_0, \dots, Q_3 \rangle \subset \mathbb{P}(V^\vee)$. Choose a linear subspace $L' \subset \mathbb{P}(V^\vee)$ of codimension 3 containing L such that L' meets W transversally in a variety Y' of dimension $2m$, and such that it meets $\mathrm{Sing}(W)$ transversally in a variety of dimension $m-2$. Set $Y = \nu^{-1}(Y')$. By construction, $Y \setminus (Y \cap \mathrm{Fix}(\sigma))$ is smooth. The symmetric bilinear form Q_i decomposes as $Q_i(x,y) = Q_i'(x) + Q_i''(y)$. The tangent space to $Y \cap \mathbb{P}_x^{m+1}$ at a point $(x,0) \in Y \cap \mathbb{P}_x^{m+1}$ is given by

$$\sum_{j=0}^{m+1} \frac{\partial Q_i'}{\partial x_j}(x,0) x_j = 0, \quad i = 1,2,3.$$

By assumption, Y intersects $\mathbb{P}_x^{m+1}$ transversally in $(x,0)$. Hence, the linear forms defining $T_{(x,0)}Y$ are independent and $Y \subset \mathbb{P}^{2m+3}$ is smooth at $(x,0)$. In a similar way we show that $Y \cap \mathbb{P}_y^{m+1}$ is smooth. $\square$

3.3 Infinitesimal Abel–Jacobi map

In this section we show that the Abel–Jacobi map associated to the family of codimension m–cycles parametrized by $\tilde{C}_t$ is non–trivial. Using a specialization argument we extend this result to general complete intersections of four quadrics.

Let $X = X_t$, $t \in U^0$ general. By Lemma 3.2.12, we can choose a smooth complete intersection $Y = \bigcap_{i=1}^{3} V(\tilde{Q}_i)$, $\tilde{Q}_i \in V$, with $Y \supset X$. The variety X corresponds to a web $M \subset \mathbb{P}(V)$, and Y corresponds to a net $N \subset M$. The intersection of $C \subset \mathcal{L}_{2m+2}(X)$ and N in M consists of a finite number of points. Choose a point $\hat{Q} \in C \setminus (C \cap N)$ such that $M = \langle \tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3, \hat{Q} \rangle$ and an $(m+2)$–plane $\Lambda \subset \hat{Q}$ such that $P = (\hat{Q}, \Lambda) \in \mathcal{L}_C(X)^0$, i.e., Z_P is smooth of dimension $m-1$. This is possible by Lemma 3.2.10. Let $\tilde{P} = g(P) \in \tilde{C}$ be the image of P under the map $g : \mathcal{L}_C(X) \to \tilde{C}$ that was obtained by Stein factorization from $f : \mathcal{L}_C(X) \to C$.

We denote the Abel–Jacobi map on X by $\psi_X : \mathrm{CH}^m_{\mathrm{hom}}(X) \to J^m(X)$. The choice of a base point $P_0 = (Q_0, \Lambda_0) \in \mathcal{L}_C(X)^0$ determines a map

$$\Phi_X : \mathcal{L}_C(X)^0 \longrightarrow J^m(X)$$

that sends P to $\psi_X(Z_P - Z_{P_0})$. Since Φ_X is constant along the fibers of $g : \mathcal{L}_C(X)^0 \to \tilde{C}$, it induces a map

$$\psi : \tilde{C} \longrightarrow J^m(X).$$

Set $Z = Z_P$. The infinitesimal Abel–Jacobi mapping $(\Phi_X)_*$ factorizes as follows:

$$T_P\mathcal{L}_C(X)^0 \xrightarrow{i_*} T_Z\mathrm{Hilb}_X^{P(x)} \xrightarrow{\sim} H^0(Z, N_{Z,X}) \xrightarrow{\Phi_*} H^m(X, \Omega_X^{m-1}).$$

The computation of the codifferential $\Phi_*^\vee$ is given by the following Lemma, which is a slight modification of a result of Welters [Wel] (see also [C2]).

Lemma 3.3.1. *The codifferential $\Phi_*^\vee$ fits into a commutative diagram with exact columns*

$$\begin{array}{ccc}
H^{m-2}(\Omega_X^{m+1} \otimes N_{X,Y}) & \xrightarrow{\gamma} & H^{m-2}(\bigwedge^{m-2} N_{Z,X} \otimes (N_{X,Y} \otimes K_X)|_Z) \\
\downarrow \delta & & \downarrow \alpha \\
H^{m-1}(\Omega_X^m) & \xrightarrow{\Phi_*^\vee} & H^{m-1}(K_Z \otimes N_{Z,X}^\vee) \\
\downarrow & & \downarrow \beta \\
H^{m-1}(\Omega_Y^{m+1} \otimes N_{X,Y}) & \longrightarrow & H^{m-1}(\bigwedge^{m-1} N_{Z,Y} \otimes K_X|_Z).
\end{array}$$

Proof: Griffiths [Gr1, Thm. 2.25] showed that

$$\begin{array}{cccc} \Phi_*^\vee : & H^{m-1}(X,\Omega_X^m) & \to & H^{m-1}(Z,K_Z\otimes N_{Z,X}^\vee) \\ & \| & & \| \\ & H^m(X,\Omega_X^{m-1})^\vee & & H^0(Z,N_{Z,X})^\vee \end{array}$$

is the composition of the restriction map

$$H^{m-1}(\Omega_X^m)\to H^{m-1}(\Omega_X^m\otimes\mathcal{O}_Z)$$

and the map

$$H^{m-1}(Z,\Omega_X^m\otimes\mathcal{O}_Z)\to H^{m-1}(Z,K_Z\otimes N_{Z,X}^\vee).$$

The latter is induced by the exact sequence

$$\bigwedge^2 N_{Z,X}^\vee\otimes\Omega_X^{m-2}\to\Omega_X^m\otimes\mathcal{O}_Z\to K_Z\otimes N_{Z,X}^\vee\to 0$$

that comes from the filtration on $\Omega_X^m\otimes\mathcal{O}_Z$ obtained by taking exterior powers in

$$0\to N_{Z,X}^\vee\to\Omega_X^1\otimes\mathcal{O}_Z\to\Omega_Z^1\to 0.$$

Consider the commutative diagram with exact rows and columns

(*)

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & T_Z & = & T_Z & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & T_X\otimes\mathcal{O}_Z & \to & T_Y\otimes\mathcal{O}_Z & \to & N_{X,Y}\otimes\mathcal{O}_Z & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & N_{Z,X} & \to & N_{Z,Y} & \to & N_{X,Y}\otimes\mathcal{O}_Z & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0. & & & & \end{array}$$

Taking exterior powers in both horizontal sequences and tensoring with $K_X\otimes\mathcal{O}_Z$, we obtain a commutative diagram of short exact sequences

$$\begin{array}{ccccc} \bigwedge^{m-1}T_X\otimes K_X|_Z & \to & \bigwedge^{m-1}T_Y\otimes K_X|_Z & \to & \bigwedge^{m-2}T_X\otimes N_{X,Y}\otimes K_X|_Z \\ \downarrow & & \downarrow & & \downarrow \\ \bigwedge^{m-1}N_{Z,X}\otimes K_X & \to & \bigwedge^{m-1}N_{Z,Y}\otimes K_X|_Z & \to & \bigwedge^{m-2}N_{Z,X}\otimes N_{X,Y}\otimes K_X. \end{array}$$

The adjunction formula shows that

$$\begin{array}{rcl} \bigwedge^{m-1}N_{Z,X}\otimes K_X\otimes\mathcal{O}_Z & \cong & K_Z\otimes N_{Z,X}^\vee \\ \bigwedge^{m-1}T_Y\otimes K_X & \cong & \bigwedge^{m-1}T_Y\otimes K_Y\otimes N_{X,Y}=\Omega_Y^{m+1}\otimes N_{X,Y}. \end{array}$$

If we compose the vertical maps in the previous commutative diagram with the restriction map $\mathcal{O}_X\to\mathcal{O}_Z$ and take the associated long exact sequences in cohomology, the assertion follows. $\square$

Lemma 3.3.2. *The map*

$$\gamma : H^{m-2}(X, \Omega_X^{m+1} \otimes N_{X,Y}) \to H^{m-2}(Z, \bigwedge^{m-2} N_{Z,X} \otimes N_{X,Y}|_Z \otimes K_X|_Z)$$

is surjective.

Proof: We argue by induction. Set $a = m - k$. From $(*)$ we obtain a commutative diagram

$$\begin{array}{ccc} H^{a-1}(\wedge^{a-1} T_X \otimes N_{X,Y}^{\otimes k} \otimes K_X) & \longrightarrow & H^a(\wedge^a T_X \otimes N_{X,Y}^{\otimes (k-1)} \otimes K_X) \\ \downarrow{\scriptstyle \gamma_{k+1}} & & \downarrow{\scriptstyle \gamma_k} \\ H^{a-1}(\wedge^{a-1} N_{Z,X} \otimes N_{X,Y}^{\otimes k} \otimes K_X) & \xrightarrow{\delta_k} & H^a(\wedge^a N_{Z,X} \otimes N_{X,Y}^{\otimes (k-1)} \otimes K_X). \end{array}$$

The map δ_k is the connecting homomorphism in the long exact cohomology sequence associated to the short exact sequence

$$0 \to \bigwedge^{m-k} N_{Z,X} \to \bigwedge^{m-k} N_{Z,Y} \to \bigwedge^{m-k-1} N_{Z,X} \otimes N_{X,Y} \to 0$$

tensored by $N_{X,Y}^{\otimes (k-1)} \otimes K_X|_Z$. To show that δ_k is surjective for $k = 2, \ldots, m-1$ it suffices to show that

$$H^{m-k}(Z, \bigwedge^{m-k} N_{Z,Y} \otimes N_{X,Y}^{\otimes (k-1)} \otimes K_X|_Z) = 0$$

for $k = 2, \ldots, m-1$. Since

$$K_X \cong \mathcal{O}_X(-2m+4), \quad N_{Z,Y} \cong \bigoplus^{m+1} \mathcal{O}_Z(1) \quad \text{and} \quad N_{X,Y} \cong \mathcal{O}_X(2),$$

it suffices to show that

$$H^{m-k}(Z, \mathcal{O}_Z(k-m+2)) = 0.$$

As $Z = V(2,2,2) \subset \Lambda$, $\Lambda \cong \mathbb{P}^{m+2}$, there is a resolution

$$0 \to \mathcal{O}_\Lambda(-6) \to \bigoplus^3 \mathcal{O}_\Lambda(-4) \to \bigoplus^3 \mathcal{O}_\Lambda(-2) \to \mathcal{O}_\Lambda \to \mathcal{O}_Z \to 0,$$

and the desired vanishing statement follows since $m-k+3 \leq m+1 < \dim \Lambda$. By induction, it suffices to show that γ_m is surjective. the varieties X and Z are complete intersections in $\mathbb{P}^{2m+3}$, hence they are projectively normal. The surjectivity of γ_m thus follows from the commutative diagram

$$\begin{array}{ccc} H^1(X, T_X) & \xrightarrow{\gamma_{m-1}} & H^1(Z, N_{Z,X}) \\ \uparrow & & \uparrow{\scriptstyle \delta_{m-1}} \\ H^0(X, \mathcal{O}_X(2)) & \xrightarrow{\gamma_m} & H^0(Z, \mathcal{O}_Z(2)). \end{array}$$

$\square$

Lemma 3.3.3. *If $\tilde{P} \in \tilde{C}$ is a general point, the map*

$$\psi_* : T_{\tilde{P}}\tilde{C} \to H^m(X, \Omega_X^{m-1})$$

is nontrivial.

Proof: By Lemma 3.2.11 the irreducible component H of $\mathrm{Hilb}_X^{P(x)}$ containing Z has dimension at least $\binom{m+1}{2} + 1$. Since $T_Z H \cong H^0(Z, N_{Z,X})$, we have $h^0(Z, N_{Z,X}) \geq \binom{m+1}{2} + 1$. As

$$\begin{aligned} H^{m-1}(Z, \textstyle\bigwedge^{m-1} N_{Z,Y} \otimes K_X \otimes \mathcal{O}_Z) &\cong \textstyle\bigoplus^{\binom{m+1}{2}} H^{m-1}(Z, \mathcal{O}_Z(-m+3)) \\ &\cong \textstyle\bigoplus^{\binom{m+1}{2}} H^{m-1}(Z, K_Z), \end{aligned}$$

it follows that $h^{m-1}(Z, \bigwedge^{m-1} N_{Z,Y} \otimes K_X \otimes \mathcal{O}_Z) = \binom{m+1}{2}$; hence the commutative diagram of Lemma 3.3.1 shows that $\ker\beta \neq (0)$. Choose a nonzero element $\hat{\xi} \in \ker\beta$. Then there exists an element

$$\hat{\zeta} \in H^{m-2}(Z, \textstyle\bigwedge^{m-2} N_{Z,X} \otimes (N_{X,Y} \otimes K_X)|_Z)$$

such that $\alpha(\hat{\zeta}) = \hat{\xi}$. By Lemma 3.3.2 there exists an element

$$\hat{\eta} \in H^{m-2}(X, \Omega_X^{m+1} \otimes N_{X,Y})$$

such that $\gamma(\hat{\eta}) = \hat{\zeta}$. As $\Phi_*^\vee(\delta(\hat{\eta})) = \alpha(\hat{\zeta}) = \hat{\xi} \neq 0$, the map $\Phi_*^\vee$ is non–trivial. Hence, if ξ is dual to $\hat{\xi}$ then $\Phi_*(\xi) \neq 0$. As we can choose ξ in such a way that $\xi = i_*(\chi) \in i_* T_P \mathcal{L}_C(X)^0$, the commutative diagram

$$\begin{array}{ccc} T_P\mathcal{L}_C(X)^0 & \xrightarrow{i_*} & H^0(Z, N_{Z,X}) \\ \big\downarrow{\scriptstyle g_*} & & \big\downarrow{\scriptstyle \Phi_*} \\ T_{\tilde{P}}\tilde{C} & \xrightarrow{\psi_*} & H^m(X, \Omega_X^{m-1}) \end{array}$$

shows that

$$\psi_*(g_*\chi) = \Phi_*(\xi) \neq 0,$$

hence ψ_* is non–trivial. □

Recall that there is a natural involution σ' on $\tilde{C}$ that interchanges the two rulings of $(m+2)$–planes. We define the mapping

$$\psi' : \tilde{C} \longrightarrow J^m(X)$$

by $\psi'(\tilde{P}) = \psi(\sigma'(\tilde{P})) - \psi(\tilde{P})$.

Lemma 3.3.4. *If $\tilde{P} \in \tilde{C}$ is a general point, then $\psi'(\tilde{P}) \neq 0$.*

Proof: As $\sigma'_* = -\mathrm{id}$, it follows that $\psi'_* = \psi_* \circ \sigma'_* - \psi_* = -2\psi_*$. Therefore the result follows from Lemma 3.3.3. □

To extend the results on non–triviality of the Abel–Jacobi map to general smooth complete intersections $X = \bigcap_{i=0}^{3} V(Q_i)$, where the Q_i need no longer be σ–invariant, we use a specialization argument.

Lemma 3.3.5. *If $X \subset \mathbb{P}^{2m+3}$ is a general smooth complete intersection of four quadrics, the Abel–Jacobi map ψ_X is non–trivial.*

Proof: Let $D_1 \subset \mathbb{P}(\tilde{V})$ be the discriminant locus; it is stratified by irreducible subsets

$$D_i = \{Q \in \mathbb{P}(\tilde{V}) : \dim \operatorname{Sing}(Q) \geq i-1\}$$

of codimension $\binom{i+1}{2}$ in $\mathbb{P}(\tilde{V})$. Let

$$J = \{(Q, s, W) \in \mathbb{P}(\tilde{V}) \times G(4, \tilde{V}) \times G(m+3, 2m+4) : Q \in M_s, W \subset V(Q)\}$$

be a correspondence with projection $p_1 : J \to \mathbb{P}(\tilde{V})$. Let $I = p_{12}(J) \subset \mathbb{P}(\tilde{V}) \times G(4, \tilde{V})$ be the incidence correspodence, and let I_0 be its restriction to $\mathbb{P}(V) \times G(4, V)$. We denote the projections on the first factor by $\pi_1 : I \to \mathbb{P}(\tilde{V})$ and $\pi_{1,0} : I_0 \to \mathbb{P}(V)$; likewise we have projections $\pi_2 : I \to G(4, \tilde{V})$ and $\pi_{2,0} : I_0 \to G(4, V)$. Set $\mathcal{M} = p_1^{-1}(D_2)$, $\mathcal{L} = \pi_1^{-1}(D_2)$ and $\mathcal{L}_0 = \pi_{1,0}^{-1}(\Delta_{1,1})$. For a quadric $Q \in \mathbb{P}(\tilde{V})$, the fiber $\pi_1^{-1}(Q)$ is isomorphic to the Grassmann variety $G(3, H)$ of 3–dimensional linear subspaces contained in a hyperplane $H \subset \tilde{V}$. Hence all the fibers of π_1 are irreducible of constant dimension, and $\mathcal{L}$ is irreducible. Choose two points $t \in T$ and $t_0 \in U$. Note that $\pi_{2,0}^{-1}(t_0) = C_{t_0}$, and that the general fiber $\pi_2^{-1}(t)$ consists of finitely many points (t, Q_i), $i = 1, \ldots, \delta$; it follows from [HT] that $\delta = \deg D_2 = \binom{2m+5}{3}$. Choose $\tau = (t, Q) \in \pi_2^{-1}(t)$ and $\tau_0 = (t_0, Q_0) \in C_{t_0}$. Since $\mathcal{L}$ is irreducible, we can connect the points τ and τ_0 by an irreducible curve γ. As D_3 has codimension 3 in D_2, we can arrange that $\gamma \cap \pi_1^{-1}(D_3) = \emptyset$.

Let $\mathcal{M} \to \tilde{\mathcal{L}} \xrightarrow{h} \mathcal{L}$ be the Stein factorization of the projection $p_{12} : \mathcal{M} \to \mathcal{L}$. Since γ does not meet the ramification locus of h, it can be lifted to a curve $\tilde{\gamma} \subset \tilde{\mathcal{L}}$ with beginpoint $\tilde{\tau}$ and endpoint $\tilde{\tau}_0 \in \tilde{C}_{t_0}$. By Lemma 3.3.4 it follows that $\psi'(\tilde{\tau}_0) \neq 0$ if $\tilde{\tau}_0$ is a general point of $\tilde{C}_{t_0}$. Set $\hat{\gamma} = \pi_2(\gamma) \subset G(4, \tilde{V})$. Let ρ be the inverse of the isomorphism $\pi_2 \circ h : \tilde{\gamma} \xrightarrow{\sim} \hat{\gamma}$. Define a normal function ν over an open subset $\hat{U} \subset \hat{\gamma}$ by $\nu(s) = \psi'_s(\rho(s))$. Since $\nu(t_0) = \psi'(\tilde{\tau}_0) \neq 0$, it follows that $\nu(t) \neq 0$ for a general point $t \in T$. $\square$

Theorem 3.3.6. *If $X = V(2,2,2,2) \subset \mathbb{P}^{2m+3}$ $(m \geq 2)$ is a very general smooth complete intersection of four quadrics, then the image of the Abel–Jacobi map $\psi_X : \mathrm{CH}^m_{\mathrm{hom}}(X) \to J^m(X)$ is not contained in the torsion points of $J^m(X)$.*

Proof: We use the specialization argument of Lemma 3.3.5. If $\tilde{P} \in \tilde{C} = \tilde{C}_{t_0}$ is a general point, the map

$$\psi'_* : T_{\tilde{P}}\tilde{C} \to H^m(X_0, \Omega^{m-1}_{X_0})$$

is non–trivial by Lemmas 3.3.3 and 3.3.4, hence ψ' is non–torsion. Set

$$\tilde{C}_n = \{\tilde{P} \in \tilde{C} : n\psi'(\tilde{P}) = 0\}.$$

By Chow's theorem, $\tilde{C}_n \subset \tilde{C}$ is a Zariski closed subset. For every $n \in \mathbb{N}$ there exists a Zariski closed subset $T_n \subset T$ such that every point $\tilde{\tau} \in h^{-1}(\pi_2^{-1}(t))$ specializes to a point $\tilde{\tau}_0 \in \tilde{C}_{t_0}$ with $\tilde{\tau}_0 \notin \tilde{C}_n$ if $t \in T \setminus T_n$, and the assertion follows. $\square$

As an immediate consequence we find that the Griffiths group $\mathrm{Griff}^m(X)$ is non–torsion:

Corollary 3.3.7. *If X is a very general smooth complete intersection of four quadrics, then* $\mathrm{Griff}^m(X) \otimes \mathbb{Q} \neq 0$.

Proof: Using the results in [DK, Exposé XI] we find that the Hodge structure on $H^{2m-1}(X)$ has level three. A standard monodromy argument then shows that $J^m_{\mathrm{alg}}(X_t) = 0$ for very general t (cf. [H1] or [Shi]). $\square$

Chapter 4

Complete intersections in Grassmann varieties

4.1 Introduction

One of the ingredients that we needed for the computations in Chapter 2 was the Bott vanishing theorem; it enabled us to interpret the variable cohomology of a smooth complete intersection in projective space in terms of the Jacobi ring. In principle, this theorem allows us to extend the results of Chapter 2 to complete intersections in arbitrary compact homogeneous Kähler manifolds. In this chapter we study the case of complete intersections in Grassmann varieties. We start by recalling some results on cohomology of homogeneous vector bundles in Section 4.2; these results are mainly included for their use in the next chapter. In Section 4.3 we generalize the main result of Chapter 2, using the abstract version of the Jacobi ring introduced in Chapter 1 and the symmetrizer lemma. As before, let $\mathrm{CH}^m(Y)_0$ denote the pullback of $H^{2m}_{\mathrm{pr}}(Y)$ under the cycle class map. The main result, Theorem 4.3.11, asserts that the image of the Abel–Jacobi map for a very general complete intersection $X = V(d_0, \ldots, d_r) \subset Y = G(s, \ell + 1)$ of odd dimension $2m - 1$ coincides (up to torsion) with the image of the composed map $\mathrm{CH}^m(Y)_0 \to \mathrm{CH}^m_{\mathrm{hom}}(X) \to J^m(X)$ if $\min(d_0, \ldots, d_r)$ is sufficiently large. The most natural way to prove this result is to consider the infinitesimal invariants of normal functions, as in Chapter 2. This approach is connected with a delicate problem; therefore we use a different method, based on Nori's results [No]. We do not obtain sharp degree bounds, as in Chapter 2, because it is hard to work out the degree conditions imposed by the Bott vanishing theorem. Therefore we investigate the case of complete intersections in Grassmann varieties of lines in $\mathbb{P}^\ell$ in more detail in Section 4.4; in this case, we obtain more precise degree bounds. We used the Maple package 'Schubert' of S. Katz and S. A. Strømme for the computation of Hodge numbers of complete intersections in Grassmann varieties.

4.2 Homogeneous vector bundles

For later use in this and the next chapter, we collect some results on homogeneous vector bundles. We refer to [FH] or [Hu] for basic facts concerning representation theory. Let G be a connected and simply connected complex Lie group, and let $P \subset G$ be a parabolic subgroup. The quotient space $Y = G/P$ is a compact homogeneous space.

Definition 4.2.1. *A holomorphic vector bundle $\pi : E \to Y$ over a homogeneous space $Y = G/P$ is called homogeneous if*

(i) There is a G–action on the total space of E such that the projection map π is G–equivariant.

(ii) The map $E_y \to E_{g.y}$ induced by the action of G is an isomorphism of vector spaces for all $g \in G$.

There is a 1–1 correspondence between homogeneous vector bundles over $Y = G/P$ and representations of P. A homogeneous vector bundle E_ρ is said to be *irreducible* if the corresponding representation $\rho : P \to W$ is irreducible. Note that $\Gamma(G/P, E_\rho) = \mathrm{Ind}_P^G(W)$ is the induced representation of G.

Let R^+ be the finite set of positive roots, and let $T \subset G$ be a maximal torus. Let B be the Borel subgroup generated by T and the negative root groups. The Killing form induces an inner product $(\,,)$ on the character group $\Lambda = \mathrm{Hom}(T, \mathbb{C}^*)$. A weight $\lambda \in \Lambda$ is called *singular* if $(\lambda, \alpha) = 0$ for some positive root $\alpha \in R^+$. If λ is not singular, it is called *regular* and we define

$$\mathrm{index}(\lambda) = \#\{\alpha \in R^+ : (\lambda, \alpha) < 0\}.$$

The cohomology groups of irreducible homogeneous vector bundles can be computed by the following theorem of Bott (see [Bott, Theorem IV']):

Theorem 4.2.2. *Let P be a parabolic subgroup of a semisimple complex Lie group G. Let W_λ be the irreducible P–module with highest weight λ and let $E_\lambda = G \times_P W_\lambda$ the corresponding homogeneous vector bundle on $Y = G/P$. Let $\delta = \sum_i \lambda_i$ be the sum of the fundamental dominant weights, and let W be the Weyl group.*

(i) If $\lambda+\delta$ is singular, then $H^p(Y,E_\lambda)=0$ for all $p\geq 0$.

(ii) If $\lambda+\delta$ is regular, then

$$H^p(Y,E_\lambda)=\begin{cases} 0 & \text{if } p\neq \operatorname{index}(\lambda+\delta)\\ \Gamma_{\mu-\delta} & \text{if } p=\operatorname{index}(\lambda+\delta),\end{cases}$$

where μ is the unique dominant weight in the W–orbit of $\lambda+\delta$ and $\Gamma_{\mu-\delta}$ denotes the irreducible G–module with highest weight $\mu-\delta$.

Bott's theorem also allows us to calculate the cohomology groups of homogeneous vector bundles that are completely reducible, i.e., vector bundles that correspond to representations of the reductive part of P.

In the remainder of this section we consider homogeneous vector bundles over Grassmann varieties. Let V be a complex vector space of dimension $\ell+1$ with basis $\{e_1,\ldots,e_{\ell+1}\}$. Set $t=\ell+1-s$. The Grassmann variety $Y=G(s,V)$ of s–dimensional linear subspaces of V is a homogeneous space of the form $Y=G/P_t$, where $G=\mathrm{SL}(\ell+1,\mathbb{C})$ and

$$P_t=\left\{\begin{pmatrix} h_1 & 0\\ h_3 & h_4\end{pmatrix} : h_1\in \mathrm{GL}(t,\mathbb{C}),\ h_4\in\mathrm{GL}(s,\mathbb{C}),\ \det(h_1).\det(h_4)=1\right\}.$$

Since $G(s,V)\cong G(\ell+1-s,V^\vee)$, we may assume that $2s\leq \ell-1$. Note that $\dim G=s(\ell+1-s)=st$.

The root system of $\mathrm{SL}(\ell+1,\mathbb{C})$ is $R=\{e_i-e_j : 1\leq i,j\leq \ell+1\}$, and the set of positive roots is $R^+=\{e_i-e_j : 1\leq i<j\leq \ell+1\}$. The set of simple roots is $\Delta=\{\alpha_1,\ldots,\alpha_\ell\}$, where $\alpha_i=e_i-e_{i+1}$. We shall sometimes denote the weight $\beta_1e_1+\ldots+\beta_{\ell+1}e_{\ell+1}$ by $(\beta_1,\ldots,\beta_{\ell+1})$. For $1\leq i\leq \ell$ we define $\lambda_i=e_1+\ldots+e_i$. The set of dominant weights is

$$\Lambda^+=\{\sum_i n_i\lambda_i, n_i\geq 0 \text{ for all } i=1,\ldots,\ell\}.$$

Thus $\beta=(\beta_1,\ldots,\beta_{\ell+1})$ is dominant if and only if $\beta_1\geq\ldots\geq\beta_{\ell+1}$. The weights $\lambda_1,\ldots,\lambda_\ell$ are called the fundamental dominant weights.

Let $T\subset \mathrm{SL}(\ell+1,\mathbb{C})$ be the subgroup of diagonal matrices, and let B be the Borel subgroup of lower triangular matrices. The reductive part of the parabolic subgroup P_t is

$$P_{\mathrm{red}}=\left\{\begin{pmatrix} h_1 & 0\\ 0 & h_4\end{pmatrix} : h_1\in \mathrm{GL}(t,\mathbb{C}),\ h_4\in\mathrm{GL}(s,\mathbb{C}),\ \det(h_1).\det(h_4)=1\right\},$$

and the semisimple part of P_t is isomorphic to $\mathrm{SL}(t,\mathbb{C})\times\mathrm{SL}(s,\mathbb{C})$. The weights of an irreducible representation of P_t are dominant for the semisimple part of P_t, i.e., they are of the form $\beta = (\beta_1, \ldots, \beta_t; \beta_{t+1}, \ldots, \beta_{\ell+1})$ with $\beta_1 \geq \ldots \geq \beta_t$ and $\beta_{t+1} \geq \ldots \geq \beta_{\ell+1}$.

The vector bundles Ω_Y^q are completely reducible. Their decomposition is given by a theorem of Kostant [Kos]. Using the theorems of Bott and Kostant, Snow has obtained a combinatorial criterion for the vanishing of the groups $H^p(Y, \Omega_Y^q(k))$ (see [Sn, §3]): let $(p_1, \ldots, p_s)$ be a partition of q with $p_1 \leq \ldots \leq p_s$, and let $(p'_1, \ldots, p'_t)$ be the conjugate partition. These data correspond to a Young diagram embedded in the $s \times t$ matrix of squares

$$N = \{(i,j) : 1 \leq i \leq s,\ 1 \leq j \leq t\},$$

where p_i is the number of squares of the Young diagram in the ith row and p'_j is the number of squares in the jth column. To each square (i,j) of the Young diagram we associate its *hook length*, i.e., the natural number

$$h_{ij} := p_i + p'_j - (s + j - i);$$

the hook length h_{ij} is the sum of the number of squares (a,j) of the Young diagram such that $a \geq i$ and the number of squares (i,b) such that $b \geq j$, where the square (i,j) itself is counted once (cf. [FH, p. 50], where a different convention is used: the diagrams consist of rows of decreasing length).

Proposition 4.2.3. (Snow)

(i) If there exists a square (i,j) with $h_{ij} = k$ for every diagram with q squares embedded in the $s \times t$ matrix of squares N, then $H^p(Y, \Omega_Y^q(k)) = 0$ for all $p \geq 0$.

(ii) Suppose that there exists a Young diagram with q squares that does not contain a square (i,j) with $h_{ij} = k$. Then $H^p(Y, \Omega_Y^q(k)) \neq 0$, where p is the number of squares (i,j) such that $h_{ij} > k$.

Example 4.2.4. Consider the Grassmann variety $Y = G(3,6)$ (s = t = 3). We take $q = 6$. The diagram

1		
3	1	
5	3	1

associated to the partition $(p_1, p_2, p_3) = (1,2,3)$ shows that $H^3(Y, \Omega_Y^6(2)) \neq 0$, $H^1(Y, \Omega_Y^6(4)) \neq 0$.

Theorem 4.2.5. (Snow) *Let $Y = G(s, \ell+1)$ be the Grassmann variety of $(s-1)$–dimensional linear subspaces of $\mathbb{P}^\ell$. Fix an integer $k \geq 1$. Then $H^p(Y, \Omega_Y^q(k)) = 0$ if one of the following conditions is satisfied:*

(a) $q > s(\ell+1-s) - s$, $Y \neq G(2,4)$.

(b) $k \geq \min(q, \ell)$, $p \geq 1$.

(b′) $k \geq \min(q, \ell) - 1$, $p \geq 2$.

(c) $sp \geq (s-1)q > 0$.

Altough this result is sharp in the generality in which it is stated, it can be improved for specific values of s and ℓ. To illustrate this, we work out the case $s = 2$ using Proposition 4.2.3. This result, previously obtained by Konno [Ko3], will be used in Section 4.4.

Lemma 4.2.6. *Let $Y = G(2, \ell+1)$ be the Grassmann variety of lines in $\mathbb{P}^\ell$. If $p \geq 1$ and $k \geq 1$ then*

$$H^p(Y, \Omega_Y^q(k)) \neq 0 \iff 3p - 1 < q < p + \ell \text{ and } k = q - 2p + 1.$$

Proof: We consider the Young diagrams associated to partitions $(p, q-p)$ with $q - \ell < p \leq q$. The only natural number $1 \leq k \leq q - p + 1$ that may not occur in the diagram is $k = q - 2p + 1$. Since there exists a square (i, j) with $h_{ij} = q - 2p + 1$ if and only if $p \geq q - 2p + 1$, the assertion follows from Proposition 4.2.3. □

4.3 Symmetrizer lemma

We consider smooth complete intersections $X = V(d_0, \ldots, d_r) \cap Y$ inside the Grassmann variety $Y = G(s, \ell+1)$ $(s \geq 2)$. As in Chapter 2, the main tool for studying the image of the Abel–Jacobi map is the symmetrizer lemma. Since Y has no cohomology in odd degree, the Lefschetz hyperplane theorem shows that the Abel–Jacobi mapping ψ_X is trivial if dim X is even. Thus we may assume that $n = \dim X = 2m-1$ is odd. Note that dim $Y = s(\ell+1-s)$.

Lemma 4.3.1. *Let $X = V(d_0, \ldots, d_r) \cap Y$ be a smooth complete intersection in Y. Fix an integer p such that $0 \leq p \leq n = \dim X$. If*

$$\sum_{i=1}^{r} d_i \geq \min(n+r-p+1, \ell) - 1 \quad (r > 0)$$
$$d_0 \geq \min(n+r-p+2, \ell) - 1 \quad (r = 0).$$

then there is an exact sequence

$$0 \to H^{n-p+1,p}_{\mathrm{pr}}(Y) \to R_{p,d(X)} \to H^{n-p,p}_{\mathrm{var}}(X) \to 0.$$

Proof: We verify the conditions (1)–(4) of Lemma 1.3.7. The desired statement then follows from Corollary 1.3.9. The strongest condition on k needed to guarantee the vanishing $H^a(Y, \Omega^b_Y(k)) = 0$ in Theorem 4.2.5 occurs if $b \leq n+r+1-s$ is chosen to be maximal with respect to a. The case $a = 1$ can only occur in condition (3) of Lemma 1.3.7, since otherwise $b = n+r > n+r+1-s$. As the conditions that (1)–(4) impose on the degrees become weaker as ν increases, it suffices to treat the case $\nu = 0$ (unless $r = 0$, in which case we take $\nu = 1$ in condition (4)). Theorem 4.2.5 shows that (4) is satisfied if the conditions of the Lemma are fulfilled, and the conditions (1)–(3) of Lemma 1.3.7 are weaker. □

Lemma 4.3.2. *The conclusion of Lemma 4.3.1 holds for all $p \geq m-2$ if*

$$\sum_{i=\min(1,r)} d_i \geq \ell - 1.$$

Proof: The strongest condition occurs if we take $p = m-2$ in Lemma 4.3.1. The conditions then read

$$\sum_{i=1}^{r} d_i \geq \min(m+r+2, \ell) - 1$$
$$d_0 \geq \min(m+3, \ell) - 1.$$

We then simply note that $m+r+2 \geq \ell$ for all $r \geq 0$:

$$\begin{aligned} m+r+2 \geq \ell \quad &\iff \quad 2m+2r+4 \geq 2\ell \\ &\iff \quad s(\ell+1-s)+r+4 \geq 2\ell \\ &\iff \quad (s-2)\ell + s(1-s) + r + 4 \geq 0, \end{aligned}$$

and since $\ell \geq 2s-1$, it suffices to show that

$$(s-2)(2s-1) + s(1-s) + r + 4 = (s-2)^2 + r + 2 \geq 0.$$

□

As in Chapter 2, we define $E = \bigoplus_{i=0}^{r} \mathcal{O}_Y(d_i)$ and denote the associated projective bundle $\mathbb{P}(E^\vee)$ by P. In Chapter 1, we defined the Jacobi ring R as the quotient of the ring

$$S = \bigoplus_{p,q\geq 0} H^0(P, K_P^{\otimes q} \otimes \xi_E^{p+1})$$

by the Jacobi ideal J. In Remark 1.3.8 we defined a natural bigrading on S and R. Using the multi–index notation from Chapter 2, we note that

$$\begin{aligned} S_{p,d(X)} &= H^0(Y, K_Y \otimes \det E \otimes S^p E) \\ &\cong \bigoplus_{I:|I|=p} H^0(Y, \mathcal{O}_Y(d(X) + \langle d, I\rangle\,)) \end{aligned}$$

where $d(X) = d_0 + \ldots + d_r - \ell - 1$.

Lemma 4.3.3. (symmetrizer lemma) *The complex*

$$\bigwedge^2 S_{1,0} \otimes R_{p-2,d(X)} \longrightarrow S_{1,0} \otimes R_{p-1,d(X)} \longrightarrow R_{p,d(X)}$$

is exact at the middle term if $p \geq 2$ and

(1) $\sum_{i=0}^{r} d_i + (p-2)d_r \geq \ell + 2$

(2) $\sum_{i=1}^{r} d_i + (p-1)d_r \geq \ell + 1.$

As in Chapter 2, a diagram chase shows that the symmetrizer lemma follows if we prove that

(i) The complex

$$\bigwedge^2 S_{1,0} \otimes S_{p-2,d(X)} \longrightarrow S_{1,0} \otimes S_{p-1,d(X)} \longrightarrow S_{p,d(X)}$$

is exact at the middle term.

(ii) The map

$$S_{1,0} \otimes J_{p-1,d(X)} \longrightarrow J_{p,d(X)}$$

is surjective.

The statements (i) and (ii) are proved in Lemmas 4.3.4 – 4.3.6. The idea for proving (i) is to relate this statement to the corresponding statement for projective space by the Plücker embedding of Y in $\mathbb{P}^N = \mathbb{P}(\wedge^s V)$; cf. [Ko1, Lemma 7.1.6]. Let $I_Y = \oplus_k I_k$ be the ideal of Y in $\mathbb{P}^N$. The Plücker embedding is projectively normal, and the ideal I_Y is generated by quadrics [HP, VII, §7, Thm. I]. Define $\tilde{E} = \bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^N}(d_i)$ and set

$$\tilde{S}_{a,b} = H^0(\mathbb{P}^N, S^a \tilde{E} \otimes \mathcal{O}_{\mathbb{P}^N}(b)).$$

Lemma 4.3.4. *The complex*

$$\bigwedge^2 S_{1,0} \otimes S_{p-2,d(X)} \to S_{1,0} \otimes S_{p-1,d(X)} \to S_{p,d(X)}$$

is exact at the middle term if condition (1) *of Lemma 4.3.3 is satisfied and* $p \geq 2$.

Proof: Consider the commutative diagram with exact columns

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\ker r_3 & \longrightarrow & \ker r_2 & \longrightarrow & \ker r_1 \\
\downarrow & & \downarrow & & \downarrow \\
\bigwedge^2 \tilde{S}_{1,0} \otimes \tilde{S}_{p-2,d(X)} & \longrightarrow & \tilde{S}_{1,0} \otimes \tilde{S}_{p-1,d(X)} & \longrightarrow & \tilde{S}_{p,d(X)} \\
\downarrow r_3 & & \downarrow r_2 & & \downarrow r_1 \\
\bigwedge^2 S_{1,0} \otimes S_{p-2,d(X)} & \longrightarrow & S_{1,0} \otimes S_{p-1,d(X)} & \longrightarrow & S_{p,d(X)} \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

where the maps r_1, r_2 and r_3 are restriction maps. The desired statement follows by a diagram chase if

(i) The complex

$$\bigwedge^2 \tilde{S}_{1,0} \otimes \tilde{S}_{p-2,d(X)} \to \tilde{S}_{1,0} \otimes \tilde{S}_{p-1,d(X)} \to \tilde{S}_{p,d(X)}$$

is exact at the middle term.

(ii) $\ker r_2 \to \ker r_1$ is surjective.

The first statement follows from Lemma 2.3.4 with $k = d(X)$ if condition (1) of Lemma 4.3.3 is satisfied; note that the ring $\tilde{S}$ was denoted by S in Chapter 2. For the second statement, we note that

$$\ker r_1 = \bigoplus\nolimits_{J:|J|=p} I_{d(X)+\langle d,J\rangle}$$

and

$$\ker r_2 \supseteq \tilde{S}_{1,0} \otimes \left(\bigoplus\nolimits_{I:|I|=p-1} I_{d(X)+\langle d,I\rangle}\right).$$

Thus it suffices to show that the map

$$\tilde{S}_{1,0} \otimes I_{d(X)+\langle d,I\rangle} \to I_{d(X)+\langle d,I+(i)\rangle}$$

is surjective for all multi–indices I with $|I| = p-1$ and for all $i \in \{0,\ldots,r\}$. Since the ideal I_Y is generated in degree two, this follows if $d(X) + \langle d, I\rangle \geq 2$ for all multi–indices I with $|I| = p-1$. Hence the condition

$$\sum_{i=0}^{r} d_i + (p-1)d_r \geq \ell + 3$$

is sufficient; it is weaker than condition (1) of Lemma 4.3.3. $\square$

Lemma 4.3.5. *The map*

$$H^0(Y, T_Y) \otimes H^0(Y, \mathcal{O}_Y(a)) \to H^0(Y, T_Y(a))$$

is surjective for all $a \geq 0$.

Proof: (cf. [Ko1]) The tangent bundle T_Y is the homogeneous vector bundle that corresponds to the adjoint representation, with highest weight $\lambda = \lambda_1 + \lambda_\ell$, of the parabolic subgroup P_t on $\mathfrak{g}/\mathfrak{p}$; see [Weh] or [Kü]. The line bundle $\mathcal{O}_Y(a)$ corresponds to the character $\chi : P \to \mathbb{C}_\mu$. Hence $\mathrm{Ad} \otimes \mu$ is the irreducible representation of P_t with highest weight $\lambda_1 + a.\lambda_t + \lambda_\ell$, and the induced representation $H^0(Y, T_Y(a))$ is the irreducible representation of $SL(\ell+1, \mathbb{C})$ with lowest weight $-\lambda_1 - a.\lambda_t - \lambda_\ell$. Since the map

$$H^0(Y, T_Y) \otimes H^0(Y, \mathcal{O}_Y(a)) \to H^0(Y, T_Y(a))$$

is a nontrivial homomorphism of $SL(\ell+1, \mathbb{C})$–modules, it must be surjective. $\square$

We investigate the second condition that is needed for the proof of the symmetrizer lemma. Let Σ be the bundle of first order differential operators on sections of the line bundle ξ_E.

Lemma 4.3.6. *The map $S_{1,0} \otimes J_{p-1,d(X)} \to J_{p,d(X)}$ is surjective if the conditions* (1) *and* (2) *of Lemma 4.3.3 are satisfied and $p \geq 2$.*

Proof: There is a commutative diagram

$$\begin{array}{ccc}
H^0(P, \xi_E) \otimes H^0(P, \Sigma \otimes K_P \otimes \xi_E^{p+r-1}) & \xrightarrow{\tilde{\mu}} & H^0(P, \Sigma \otimes K_P \otimes \xi_E^{p+r}) \\
\downarrow & & \downarrow \\
S_{1,0} \otimes J_{p-1,d(X)} & \xrightarrow{\mu} & J_{p,d(X)}
\end{array}$$

where the vertical arrows are surjective by definition. Thus it suffices to show that the map $\tilde{\mu}$ is surjective. To this end, consider the commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
S_{1,0} \otimes H^0(P, K_P \otimes \xi_E^{p+r-1}) & \xrightarrow{\mu_1} & H^0(P, K_P \otimes \xi_E^{p+r}) \\
\downarrow & & \downarrow \\
S_{1,0} \otimes H^0(P, \Sigma \otimes K_P \otimes \xi_E^{p+r-1}) & \xrightarrow{\tilde{\mu}} & H^0(P, \Sigma \otimes K_P \otimes \xi_E^{p+r}) \\
\downarrow & & \downarrow \\
S_{1,0} \otimes H^0(P, \Omega_P^{n+2r} \otimes \xi_E^{p+r-1}) & \xrightarrow{\mu_2} & H^0(P, \Omega_P^{n+2r} \otimes \xi_E^{p+r}) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

that is induced by the exact sequence $0 \to \mathcal{O}_P \to \Sigma \to T_P \to 0$. A diagram chase shows that μ is surjective if the maps μ_1 and μ_2 are surjective. The map μ_1 is surjective if and only if

$$H^0(Y,E) \otimes H^0(Y, K_Y \otimes \det E \otimes S^{p-2}E) \to H^0(Y, K_Y \otimes \det E \otimes S^{p-1}E)$$

is surjective. This follows if $d(X)+(p-2)d_r \geq 0$, and this condition is weaker than condition (1) of Lemma 4.3.3. The exact sequence of tangent bundles

$$0 \to T_v \to T_P \to \pi^* T_Y \to 0$$

induces a short exact sequence

$$0 \to \pi^* K_Y \otimes \Omega_v^{r-1} \to \Omega_P^{n+2r} \to \pi^*\Omega_Y^{n+r} \otimes \Omega_v^r \to 0, \tag{4.1}$$

and from the relative Euler sequence (1.6) we obtain a short exact sequence

$$0 \to \pi^* {\bigwedge}^{r+1} E \otimes \xi_E^{-r-1} \to \pi^* {\bigwedge}^r E \otimes \xi_E^{-r} \to \Omega_v^{r-1} \to 0.$$

Hence

$$H^1(P, \pi^* K_Y \otimes \Omega_v^{r-1} \otimes \xi_E^{p+r-1}) = H^1(P, \pi^* K_Y \otimes \Omega_v^{r-1} \otimes \xi_E^{p+r}) = 0,$$

and from the sequence 4.1 we obtain a commutative diagram with exact columns

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
S_{1,0} \otimes H^0(P, \pi^* K_Y \otimes \Omega_v^{r-1} \otimes \xi_E^{p+r-1}) & \xrightarrow{\nu_1} & H^0(P, \pi^* K_Y \otimes \Omega_v^{r-1} \otimes \xi_E^{p+r}) \\
\downarrow & & \downarrow \\
S_{1,0} \otimes H^0(P, \Omega_P^{n+2r} \otimes \xi_E^{p+r-1}) & \xrightarrow{\mu_2} & H^0(P, \Omega_P^{n+2r} \otimes \xi_E^{p+r}) \\
\downarrow & & \downarrow \\
S_{1,0} \otimes H^0(P, \pi^*\Omega_Y^{n+r} \otimes \Omega_v^r \otimes \xi_E^{p+r-1}) & \xrightarrow{\nu_2} & H^0(P, \pi^*\Omega_Y^{n+r} \otimes \Omega_\otimes^r \xi_E^{p+r}) \\
\downarrow & & \downarrow \\
0 & & 0.
\end{array}$$

This diagram shows that μ_2 is surjective if ν_1 and ν_2 are surjective. The map ν_1 is surjective if

$$H^0(Y,E) \otimes H^0(Y, K_Y \otimes {\bigwedge}^r E \otimes S^{p-1}E) \to H^0(Y, K_Y \otimes {\bigwedge}^r E \otimes S^p E)$$

is surjective; this follows if condition (2) of Lemma 4.3.3 is satisfied. The map ν_2 is surjective if and only if

$$H^0(Y,E) \otimes H^0(Y, \Omega_Y^{n+r} \otimes \det E \otimes S^{p-2}E) \to H^0(Y, \Omega_Y^{n+r} \otimes \det E \otimes S^{p-1}E)$$

is surjective. As $\Omega_Y^{n+r} = T_Y \otimes K_Y$, this follows from Lemma 4.3.5 if condition (1) of Lemma 4.3.3 is satisfied. $\square$

Lemma 4.3.7. *Suppose that the conditions (1)–(4) of Lemma 1.3.7 are satisfied for $b-1 \le p \le b+1$. Then there is a commutative diagram*

$$\begin{array}{ccccc} H^{2m-b}(\Omega^b_{Y,X_t}) & \to & \Omega^1_{U,t} \otimes H^{2m-b+1}(\Omega^{b-1}_{Y,X_t}) & \to & \Omega^2_{U,t} \otimes H^{2m-b+2}(\Omega^{b-2}_{Y,X_t}) \\ \downarrow & & \downarrow & & \downarrow \\ R^\vee_{b,d(X)} & \to & S^\vee_{1,0} \otimes R^\vee_{b-1,d(X)} & \to & \bigwedge^2 S^\vee_{1,0} \otimes R^\vee_{b-2,d(X)} \end{array}$$

where the verical arrows are isomorphisms.

Proof: We shall only give a brief outline of the proof, and leave the details to the reader. We have a commutative diagram

$$\begin{array}{ccc} H^{2m-b}(\Omega^b_{Y,X_t}) & \longrightarrow & H^{2m-b+r}(\Omega^{b+r}_{P,\mathcal{X}_t}) \\ \downarrow & & \downarrow \\ \Omega^1_{T,t} \otimes H^{2m-b+1}(\Omega^{b-1}_{Y,X_t}) & \longrightarrow & \Omega^1_{T,t} \otimes H^{2m-b+r+1}(\Omega^{b+r-1}_{P,\mathcal{X}_t}) \\ \downarrow & & \downarrow \\ \Omega^2_{T,t} \otimes H^{2m-b+2}(\Omega^{b-2}_{Y,X_t}) & \longrightarrow & \Omega^2_{T,t} \otimes H^{2m-b+r+2}(\Omega^{b+r-2}_{P,\mathcal{X}_t}). \end{array}$$

Using the duality between $H^i(P,\Omega^j_{P,\mathcal{X}_t})$ and $H^{n+2r+1-i}(P,\Omega_P^{n+2r+1-j}(\log \mathcal{X}_t))$ (cf. Remark 1.2.3), it suffices to show that the diagram

$$\begin{array}{ccc} \wedge^2 S_{1,0} \otimes R_{b-2,d(X)} & \xrightarrow{\mathrm{id}\otimes \overline{f}_{b+r-2}} & \wedge^2 T_t \otimes H^{b+r-2}(\Omega_P^{2m-b+r+2}(\log \mathcal{X}_t)) \\ \downarrow{\scriptstyle a_2} & & \downarrow{\scriptstyle d_2} \\ S_{1,0} \otimes R_{b-1,d(X)} & \xrightarrow{\mathrm{id}\otimes \overline{f}_{b+r-1}} & T_t \otimes H^{b+r-1}(\Omega_P^{2m-b+r+1}(\log \mathcal{X}_t)) \\ \downarrow{\scriptstyle a_1} & & \downarrow{\scriptstyle d_1} \\ R_{b,d(X)} & \xrightarrow{\mathrm{id}\otimes \overline{f}_{b+r}} & H^{b+r}(\Omega_P^{2m-b+r}(\log \mathcal{X}_t)) \end{array}$$

is commutative; the maps δ_2 and δ_1 are given by cup product with the logarithmic Kodaira–Spencer class $\rho \in H^1(P, T_P(-\log \mathcal{X}_t))$. The maps

$$\overline{f}_{p+r} : R_{p,d(X)} \to H^{p+r}(\Omega_P^{n+r-p+1}(\log \mathcal{X}_t))$$

are induced by maps

$$f_{p+r} : H^0(\Omega_P^{n+2r+1}(\log \mathcal{X}_t) \otimes \xi_E^{p+r}) \to H^{p+r}(\Omega_P^{n+r-p+1}(\log \mathcal{X}_t)).$$

Let $e \in H^1(P, T_P(-\log \mathcal{X}_t) \otimes \xi_E^{-1})$ be the extension class of the short exact sequence

$$0 \to T_P(-\log \mathcal{X}_t) \to \Sigma \to \xi_E \to 0$$

from Lemma 1.2.7. The logarithmic Kodaira–Spencer map

$$T_t \to H^1(P, T_P(-\log \mathcal{X}_t))$$

can be identified with the connecting homomorphism

$$H^0(P, \xi_E) \to H^1(P, T_P(-\log \mathcal{X}_t))$$

associated to the exact sequence (4.2), and the map f_{p+r} is given by cup product with e^{p+r}. Therefore the commutativity of the upper square follows, because

$$\begin{aligned}(\mathrm{id} \otimes f_{b+r-1})(a_2(\alpha_1 \wedge \alpha_2 \otimes \beta)) &= (\mathrm{id} \otimes f_{b+r-1})(\alpha_2 \otimes (\alpha_1\beta) - \alpha_1 \otimes (\alpha_2\beta)) \\ &= \alpha_2 \otimes \alpha_1 \cup \beta \cup e^{b+r-1} \\ &\quad -\alpha_1 \otimes \alpha_2 \cup \beta \cup e^{b+r-1} \\ &= (\alpha_2 \otimes \alpha_1 \cup \beta - \alpha_1 \otimes \alpha_2 \cup \beta) \cup e^{b+r-1}\end{aligned}$$

and

$$\begin{aligned}d_2(\mathrm{id} \otimes f_{b+r-2})(\alpha_1 \wedge \alpha_2 \otimes \beta) &= d_2(\alpha_1 \wedge \alpha_2 \otimes (\beta \cup e^{b+r-2})) \\ &= \alpha_2 \otimes \alpha_1 \cup e \cup \beta \cup e^{b+r-2} \\ &\quad -\alpha_1 \otimes \alpha_2 \cup e \cup \beta \cup e^{b+r-2} \\ &= (\alpha_2 \otimes \alpha_1 \cup \beta - \alpha_1 \otimes \alpha_2 \cup \beta) \cup e^{b+r-1}.\end{aligned}$$

One can check in a similar way that the lower square commutes. The commutativity of the diagram with maps $\mathrm{id} \otimes \overline{f}_p$ follows by passage to the quotient.

□

Remark 4.3.8. The same result holds if we replace U by T, where $g : T \to U$ is a finite étale covering.

To investigate the image of the Abel–Jacobi map for very general complete intersections $X = V(d_0, \ldots, d_r) \subset Y$, we study normal functions ν that are obtained by spreading out a cycle $Z \in Z^m_{\mathrm{hom}}(X)$. Our purpose is to show that, modulo torsion, such a normal function is induced by a codimension m cycle on Y. We could proceed as in Chapter 2 and show that the Koszul cohomology group to which $\delta\nu$ belongs is isomorphic to $H^{m,m}_{\mathrm{pr}}(Y)$, but then it remains to show that $\delta\nu$ comes from a primitive Hodge class on Y. To circumvent this problem we shall use a different approach, based on results of Nori, Green and Müller–Stach (see [No], [GM]). Let $U = \mathbb{P}H^0(Y, E) \setminus \Delta$ be the complement of the discriminant locus, and let $X_U \subset Y_U = Y \times U$ be the universal family of smooth complete intersections in Y of multidegree $(d_0, \ldots, d_r)$. Let $g : T \to U$ be a finite étale covering. By base change we obtain a commutative diagram

$$\begin{array}{ccccc} X_T & \longrightarrow & X_U & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ T & \xrightarrow{g} & U & \longrightarrow & \operatorname{Spec}\mathbb{C}. \end{array}$$

Nori's idea is to extend the result of Green and Voisin by comparing the cohomology in degree $2m$ of the families Y_T and X_T.

Lemma 4.3.9. *If*

(1) $\sum_{i=0}^{r} d_i + (m-2)d_r \geq \ell + 2$

(2) $\sum_{i=1}^{r} d_i + (m-1)d_r \geq \ell + 1$

(3) $\sum_{i=\min(1,r)} d_i \geq \ell - 1$

then $H^{2m}(Y_T, \mathbb{Q}) \cong H^{2m}(X_T, \mathbb{Q})$.

Proof: We show that

$$H^{2m}(Y_T, X_T; \mathbb{Q}) = H^{2m+1}(Y_T, X_T; \mathbb{Q}) = 0.$$

We give a brief outline of the main steps of the proof; details can be found in [No] or [G4, Lecture 8]. The cohomology groups $H^k(Y_T, X_T)$ carry a MHS such that $\mathrm{Gr}_p^W H^k(Y_T, X_T) = 0$ for all $p < k-1$. Hence it suffices to show that

$$F^m H^{2m}(Y_T, X_T) = F^m H^{2m+1}(Y_T, X_T) = 0.$$

These statements follow if $H^a(Y \times \{t\}, \Omega^b_{Y_T, X_T}|_{Y \times \{t\}}) = 0$ for all a and b such that $b \geq m$ and $a + b \leq 2m+1$. There is a spectral sequence that begins with

$$E_1^{p,q}(b) = \Omega^p_{T,t} \otimes H^{p+q}(Y, \Omega^{b-p}_{Y,X_t})$$

and that converges to $H^{p+q}(Y \times \{t\}, \Omega^b_{Y_T, X_T}|_{Y \times \{t\}})$; cf. [No, Remark 3.10]. Therefore we are done if we show that $E_\infty^{p,q}(b) = 0$ for all integers p, q and b such that $b \geq m$ and $p + q + b \leq 2m+1$. By the Lefschetz hyperplane theorem we may assume that $b + q \geq 2m$. Thus we are left with the terms $E_\infty^{0,2m-b}(b)$, $E_\infty^{1,2m-b}(b)$ and $E_\infty^{0,2m-b+1}(b)$ for $b \geq m$. As

$$E_1^{0,2m-b+1}(b) = H^{2m-1}(Y, \Omega^b_{Y,X_t}) \cong \ker H^{2m-b+1}(Y, \Omega^b_Y) \to H^{2m-b+1}(X_t, \Omega^b_{X_t})$$

and the Grassmann variety Y has no cohomology in odd degree, the term $E_1^{0,2m-b+1}(b)$ vanishes. The terms $E_2^{0,2m-b}(b)$ and $E_2^{1,2m-b}(b)$ are the cohomology groups at the middle term of the complexes

$$0 \to H^{2m-b}(\Omega^b_{Y,X_t}) \to \Omega^1_{T,t} \otimes H^{2m-b+1}(\Omega^{b-1}_{Y,X_t})$$

and

$$H^{2m-b}(\Omega^b_{Y,X_t}) \to \Omega^1_{T,t} \otimes H^{2m-b+1}(\Omega^{b-1}_{Y,X_t}) \to \Omega^2_{T,t} \otimes H^{2m-b+2}(\Omega^{b-2}_{Y,X_t}).$$

Set $E_1^{-p,-q}(b) = E_1^{p,q}(b)^\vee$. If condition (3) is satisfied, then Lemmas 4.3.1, 4.3.2, 4.3.7 and Remark 4.3.8 show that the dual complexes fit into commutative diagrams

$$\begin{array}{ccccc}
E_1^{-1,b-2m}(b) & \longrightarrow & E_1^{0,b-2m}(b) & \longrightarrow & 0 \\
\downarrow\cong & & \downarrow\cong & & \\
\bigwedge^2 S_{1,0} \otimes R_{b-1,d(X)} & \longrightarrow & R_{b,d(X)} & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccc}
E_1^{-2,b-2m}(b) & \longrightarrow & E_1^{-1,b-2m}(b) & \longrightarrow & E_1^{0,b-2m}(b) \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
\bigwedge^2 S_{1,0} \otimes R_{b-2,d(X)} & \longrightarrow & S_{1,0} \otimes R_{b-1,d(X)} & \longrightarrow & R_{b,d(X)}.
\end{array}$$

The vanishing of the terms $E_2^{0,2m-b}(b)$ and $E_2^{-1,2m-b}(b)$ follows from Lemma 4.3.3 if the conditions (1) and (2) are satisfied. $\square$

Recall that Deligne–Beilinson cohomology groups are defined for smooth, quasi–projective varieties, using a good compactification and the truncated complex of differential forms with logarithmic poles along the boundary divisor; see [EV].

Corollary 4.3.10. *If the conditions* (1)–(3) *of Lemma 4.3.9 are satisfied, then*

$$H^{2m}_{\mathcal{D}}(Y_T, \mathbb{Q}) \cong H^{2m}_{\mathcal{D}}(X_T, \mathbb{Q}).$$

Proof: This follows from Lemma 4.3.9 by the long exact sequence for Deligne–Beilinson cohomology [EV] and the five lemma. $\square$

Theorem 4.3.11. *Let $X = V(d_0, \ldots, d_r)$ $(d_0 \geq \ldots \geq d_r)$ be a smooth complete intersection of odd dimension $2m-1$ $(m \geq 2)$ in the Grassmann variety $Y = G(s, \ell+1)$, and let $i : X \to Y$ be the inclusion map. If X is very general and*

(1) $\sum_{i=0}^{r} d_i + (m-2)d_r \geq \ell + 2$

(2) $\sum_{i=1}^{r} d_i + (m-1)d_r \geq \ell + 1$

(3) $\sum_{i=\min(1,r)} d_i \geq \ell - 1$

then the image of

$$\mathrm{cl}_{\mathcal{D},X} : \mathrm{CH}^m(X)_{\mathbb{Q}} \to H^{2m}_{\mathcal{D}}(X,\mathbb{Q})$$

coincides with the image of

$$i^* \circ \mathrm{cl}_{\mathcal{D},Y} : \mathrm{CH}^m(Y)_{\mathbb{Q}} \to H^{2m}_{\mathcal{D}}(Y,\mathbb{Q}).$$

Proof: If $t \in U$ is a very general point, then every cycle $Z_t \in CH^m_{\mathrm{hom}}(X_t)$ can be spread out. This means that there exist a finite étale morphism $g : T \to U$, a relative cycle $Z_T \in Z^m_{\mathrm{hom}}(X_T/T)$ and a point $t_0 \in g^{-1}(t)$ such that the fiber of Z_T over t_0 equals Z_t. By Corollary 4.3.10, the Deligne cycle class $\mathrm{cl}_{\mathcal{D}}(Z_T) \in H^{2m}_{\mathcal{D}}(X_T,\mathbb{Q})$ can be lifted to an element $\alpha \in H^{2m}_{\mathcal{D}}(Y_T,\mathbb{Q})$. Let $j_t : Y \to Y_T$ be the inclusion map that sends y to (y,t), and let $p_Y : Y_T \to Y$ be the projection onto the first factor. Since $j_t^*\alpha \in H^{2m}_{\mathcal{D}}(Y,\mathbb{Q}) \cong \mathrm{Hdg}^m(Y)_{\mathbb{Q}}$ is independent of t, there exists an element $\beta \in \mathrm{Hdg}^m(Y)_{\mathbb{Q}}$ such that $p_Y^*\beta = \alpha$. As the cohomology ring of Grassmann varieties is generated by algebraic cycles, it follows that $\beta = \mathrm{cl}_Y(Z)$ for some cycle $Z \in Z^m(Y)$; by construction we have $Z \in Z^m(Y)_0$, i.e. $\mathrm{cl}_Y(Z) \in H^{2m}_{\mathrm{pr}}(Y)$. Let $k_t : X_t \to X_T$ be the inclusion map. Combining Corollary 4.3.10 with the commutative diagram

$$\begin{array}{ccc}
H^{2m}_{\mathcal{D}}(Y_T,\mathbb{Q}) & \longrightarrow & H^{2m}_{\mathcal{D}}(X_T,\mathbb{Q}) \\
\uparrow{\scriptstyle p_Y^*} & & \downarrow{\scriptstyle k_t^*} \\
H^{2m}_{\mathcal{D}}(Y,\mathbb{Q}) & \xrightarrow{i_t^*} & H^{2m}_{\mathcal{D}}(X_t,\mathbb{Q}),
\end{array}$$

we find that

$$\begin{aligned}
\mathrm{cl}_{\mathcal{D}}(Z_t) &= k_t^*\,\mathrm{cl}_{\mathcal{D}}(Z_T) \\
&= i_t^*\,\mathrm{cl}_{\mathcal{D},Y}(Z).
\end{aligned}$$

□

Remark 4.3.12. Green and Müller–Stach are able to produce the desired cycle $Z \in Z^m(Y)$ by an induction process that involves Lefschetz pencils, under the assumption that the generalized Hodge conjecture holds for the intermediate Jacobian $J^m(Y)$; see [GM] or [G4, Lecture 8]. Here we have presented a similar result in a much simpler case, where the Hodge conjecture for Y is known, but with effective degree bounds. Note that condition (3) implies the conditions (1) and (2) if $m > 2$.

4.4 Examples

We present more precise versions of Theorem 4.3.11 in two special cases: the case where $Y = G(2, \ell+1)$ is the Grassmann variety of lines in $\mathbb{P}^\ell$, and the case where $Y = G(3,6)$ is the Grassmann variety of planes in $\mathbb{P}^5$. First we consider a complete intersection $X = V(d_0, \ldots, d_r)$ of odd dimension in the Grassmann variety $Y = G(2, \ell+1)$ of lines in $\mathbb{P}^\ell$. For a nonnegative integer ν and a multi–index $I = (i_1, \ldots, i_\nu)$ we write

$$\begin{aligned} r_1(\nu, I) &= \sum_{i=0}^{r}(d_i - 1) + \sum_{k=1}^{\nu}(d_{i_k} - 1) \\ r_2(\nu, I) &= \sum_{i=1}^{r}(d_i - 1) + \sum_{k=1}^{\nu}(d_{i_k} - 1). \end{aligned}$$

Lemma 4.4.1. *If $n = \dim X = 2m-1$ and $p = m - c$ ($c \in \mathbb{Z}$) then $H^{n+1-p}(Y, \Omega^p_{Y,X}) \cong R^\vee_{p,d(X)}$, except possibly in one of the following cases:*

(i) ℓ is odd and $r + 2r_1(\nu, I) = 2c$ for some integer ν, $0 \le \nu \le p-1$, and some multi–index $I = (i_1, \ldots, i_\nu)$.

(ii) $r + 2r_1(\nu, I) = 2c+2$ for some integer ν, $0 \le \nu \le p-2$, and some multi–index $I = (i_1, \ldots, i_\nu)$.

(iii) ℓ is odd and $r + 2r_2(\nu, I) = 2c+2$ for some integer ν, $\max(0, 1-r) \le \nu \le p-1$, and some multi–index $I = (i_1, \ldots, i_\nu)$.

Proof: Using Lemma 4.2.6, we verify the conditions (1)–(4) of Lemma 1.3.7. Let us verify that condition (1) is satisfied, unless we are in case (i). We have to show that

$$H^{p-\nu}(Y, \Omega_Y^{n+r+1-p+\nu} \otimes \det E \otimes S^\nu E) = 0 \text{ for all } 0 \le \nu \le p-1.$$

Lemma 4.2.6 shows that this statement fails if and only if there exists an integer ν in the indicated range and a multi–index $I = (i_1, \ldots, i_\nu)$ such that

$$3(p-\nu) - 1 < 2\ell - 2 - (p-\nu) < (p-\nu) + \ell \tag{4.2}$$

and

$$r + \nu + 1 + r_1(\nu, I) = 2\ell - 2 - 3(p-\nu) + 1. \tag{4.3}$$

Rewriting (4.2), we obtain $2\ell - 4 < 4(p-\nu) < 2\ell - 1$. This inequality can only hold if ℓ is odd and $4(p-\nu) = 2\ell - 2$. Suppose that (4.2) holds. Then $2\nu = 2p - \ell + 1 = 2m - 2c - \ell + 1$. The equality (4.3) holds if and only if

$$r + r_1(\nu, I) = \ell - 1 - 3p + 2m - 2c.$$

Multiplying this equation by two and using $\dim Y = 2\ell - 2 = 2m + r$, we obtain

$$r + 2r_1(\nu, I) = 2c.$$

In a similar way we show that condition (2) is satisfied unless we are in case (ii) with ℓ odd, that condition (3) is satisfied unless we are in case (ii) with ℓ even and that condition (4) is satisfied unless we are in case (iii). $\square$

Proposition 4.4.2. *Let $X = V(d_0, \ldots, d_r)$ be a very general complete intersection of odd dimension $2m-1$ ($m \geq 2$, $d_0 \geq \ldots \geq d_r$) in the Grassmann variety $Y = G(2, \ell+1)$ of lines in $\mathbb{P}^\ell$. Then the conclusion of Theorem 4.3.11 holds, except possibly in one of the following cases:*

1. $r = 0$

$$\begin{aligned} X &= V(1) \subset G(2, \ell+1),\ \ell \geq 3 \\ X &= V(2) \subset G(2, \ell+1),\ 3 \leq \ell \leq 7, \ell = 11 \\ X &= V(3) \subset G(2, \ell+1),\ \ell = 3, \ell = 5, \ell = 7 \\ X &= V(4) \subset G(2, \ell+1),\ \ell = 3, \ell = 7 \end{aligned}$$

2. $r = 2$

$$\begin{aligned} X &= V(d, 1, 1) \subset G(2, \ell+1),\ \ell \geq 4, d \geq 1 \\ X &= V(d, 2, 1) \subset G(2, \ell+1),\ \ell \geq 4, d \geq 2 \\ X &= V(d, 2, 2) \subset G(2, 6) \end{aligned}$$

3. $r = 4$

$$X = V(d, 1, 1, 1, 1) \subset G(2, \ell+1),\ \ell \geq 5$$

4. $r = 6$

$$X = V(1, 1, 1, 1, 1, 1, 1) \subset G(2, \ell+1),\ \ell \geq 6.$$

Proof: Argue as in the proof of Theorem 4.3.11, using Lemma 4.4.1 and Lemma 4.3.3. $\square$

Remark 4.4.3. I do not know whether the degree bounds in Proposition 4.4.2 are sharp. The following cases are genuine counterexamples:

(i) $X = V(1) \subset G(2, \ell+1),\ \ell \geq 3$, m even

(ii) $X = V(1,1,1) \subset G(2,\ell+1)$, ℓ even, m even.

(iii) $X = V(d) \subset G(2,4)$, $2 \leq d \leq 4$

(iv) $X = V(2,1,1) \subset G(2,5)$

(v) $X = V(1,1,1,1,1) \subset G(2,6)$

(vi) $X = V(1,1,1,1,1,1,1) \subset G(2,7)$

(vii) $X = V(d,1,1) \subset G(2,\ell+1)$, ℓ odd

(viii) $X = V(2) \subset G(2,5)$

For the cases (i) and (ii) this simply follows because $H^{2m-1}(X) = 0$, whereas $\mathrm{Hdg}^m(Y)_{\mathrm{pr}} \otimes \mathbb{Q} \neq 0$. Note that the conclusion of Proposition 4.4.2 trivially holds if m is odd, since in this case both $H^{2m-1}(X)$ and $\mathrm{Hdg}^m(Y)_{\mathrm{pr}} \otimes \mathbb{Q}$ are zero. In case (iii) the Grassmann variety Y is a smooth quadric in $\mathbb{P}^5$; its hypersurface sections of degree two and three are Fano threefolds, and it is known that in these cases the intermediate Jacobian is covered by families of 1–cycles (see [Ty1]). The hypersurface $X = V(4) \subset G(2,4)$ is a Calabi–Yau threefold; Paranjape has shown that its Abel–Jacobi image is not finitely generated [Par]. The varieties appearing in (iv) and (v) are Fano threefolds, and their intermediate Jacobians are again covered by families of 1–cycles (see [Pu]). In case (vi) we have a Calabi–Yau threefold; since $\mathrm{Hdg}^3_{\mathrm{pr}}(G(2,7)) = 0$, we obtain a counterexample using a result of Voisin [V6]. Case (vii) is treated using a result of Donagi [Don]; he proved that if $X' = V(1,1) \subset G(2,\ell+1)$, ℓ odd, then $\dim_{\mathbb{Q}} \mathrm{Hdg}^m_{\mathrm{var}}(X') \otimes \mathbb{Q} \neq 0$, hence the hypersurface sections of X' are counterexamples by results of N. Katz and Zucker (cf. Remark 2.4.2 (ii)). Finally, the variety appearing in case (viii) is a Fano fivefold of index three such that $H^5(X)$ carries a Hodge structure of level one. In Chapter 5 we shall show that the generalized Hodge conjecture holds in this case.

Some possible counterexamples are

1. $X = V(1,1,1,1,1) \subset G(2,8)$
2. $X = V(1,1,1,1,1) \subset G(2,\ell+1)$, ℓ even
3. $X = V(d,1,1,1,1) \subset G(2,\ell+1)$, ℓ odd.

In the first two cases the generalized Hodge conjecture predicts that we have Jacobi inversion, as $H^{2m-1}(X)$ carries a Hodge structure of level one. The varieties $X' = V(1,1,1,1) \subset G(2,\ell+1)$ may have nontrivial Hodge classes, as $H^{2m}(X)$ carries a Hodge structure of level zero.

As a second example, we investigate a Grassmann variety of odd dimension.

Proposition 4.4.4. *Let $Y = G(3,6)$ be the Grassmann variety of planes in $\mathbb{P}^5$. If $p \geq 1$ and $k > 0$ then*

$$H^p(Y, \Omega_Y^q(k)) \neq 0$$

if and only if (p, q, k) is one of the following triples:

$$(1,3,2), (1,4,3), (1,5,3), (1,5,4), (1,6,4), (3,6,2).$$

Proof: Apply Proposition 4.2.3. □

Proposition 4.4.5. *The conclusion of Theorem 4.3.11 holds for very general complete intersections in $Y = G(3,6)$, except possibly in one of the following cases:*

1. *$X = V(d,1)$, $d \geq 1$*
2. *$X = V(d,2)$, $d \geq 2$*
3. *$X = V(d,1,1,1)$, $d \geq 1$*
4. *$X = V(1,1,1,1,1,1)$.*

Proof: Similar to the proof of Theorem 4.3.11, where we use Lemma 4.4.4 instead of of Lemma 4.3.1. □

Remark 4.4.6. Using the result in Remark 2.4.2 (ii), we conclude that the first case provides a counterexample to the conclusion of Theorem 4.3.11, since a general hyperplane section $X' = V(1) \subset G(3,6)$ has nontrivial Hodge classes [Don]. The third case is a possible counterexample, as $H^6(X')$ carries a Hodge structure of level zero if $X' = V(1,1,1) \subset G(3,6)$. In the fourth case, X is a Calabi–Yau threefold; we obtain a counterexample by [V6] (note that $H^4_{\mathrm{pr}}(G(3,6)) = 0$). The second exceptional case arises from the degree conditions needed for the Jacobi ring description of $H^7_{\mathrm{var}}(X)$, and I do not know whether it gives rise to counterexamples.

Chapter 5

The five–dimensional quadratic line complex

5.1 Introduction

The image of the Abel–Jacobi map for very general complete intersections in projective space has been studied extensively. For complete intersections in Grassmann varieties $G(s, \ell+1)$ ($s \geq 2$), much less is known. It has been proved that the Abel–Jacobi map $\psi_X : \mathrm{CH}^2_{\mathrm{hom}}(X) \to J^2(X)$ is surjective for the Fano threefolds $X = V(2,1,1) \subset G(2,5)$ and $X = V(1,1,1,1,1) \subset G(2,6)$ [Pu]. Donagi [Don] has proved the surjectivity of the Abel–Jacobi maps $\psi_X : \mathrm{CH}^{\ell-2}_{\mathrm{hom}}(X) \to J^{\ell-2}(X)$, $X = V(1,1,1) \subset G(2, \ell+1)$ (ℓ odd, $\ell \geq 5$) and $\psi_X : \mathrm{CH}^4_{\mathrm{hom}}(X) \to J^4(X)$, $X = V(1,1) \subset G(3,6)$. Motivated by Theorem 4.3.11, we shall verify that $\psi_X : \mathrm{CH}^3_{\mathrm{hom}}(X) \to J^3(X)$ is surjective for the general quadratic line complex $X = V(2) \subset G(2,5)$.

The quadratic line complex in $G(2,4)$ has been studied by numerous authors; see for instance [GH1, Chapter 6]. The quadratic line complex in $G(2,5)$ was studied by B. Segre [Se]; according to L. Roth, it is a rational variety [Ro]. A smooth quadratic line complex $X = V(2) \subset G(2,5)$ is a Fano fivefold of index 3. The cohomology group $H^5(X)$ carries a Hodge structure of level one with $h^{2,3}(X) = 10$. In Section 5.2 we show that a general five–dimensional quadratic line complex contains a family of planes, parametrized by a smooth and irreducible curve C. Using the infinitesimal Abel–Jacobi mapping associated to this family, we verify in Section 5.3 that the map $J(C) \to J^3(X)$ is nontrivial; the surjectivity of this map then follows by a monodromy argument.

5.2 The family of planes

Let V be a complex vector space of dimension 5, and let $G = G(2, V)$ be the Grassmann variety of lines in $\mathbb{P}^4 = \mathbb{P}(V)$. The variety G is embedded as a smooth six–dimensional subvariety of degree 5 in $\mathbb{P}^9 = \mathbb{P}(\wedge^2 V)$ by the Plücker embedding. We denote the line in $\mathbb{P}^4$ corresponding to a point $x \in G$ by ℓ_x. A quadratic line complex in G is the intersection of G with a quadric $Q \subset \mathbb{P}^9$; it corresponds to a five–dimensional family of lines in $\mathbb{P}^4$.

Let $p \in \mathbb{P}(V)$ be a point, and let $\sigma(p) = \{x \in G : p \in \ell_x\}$ be the corresponding Schubert cycle. Since the tangent space $T_x G$ is spanned by

$$T_x G \cap G = \{z \in G : \ell_z \cap \ell_x \neq \emptyset\} = \cup_{p \in \ell_x} \sigma(p),$$

the line spanned by two points x, $y \in G$ is contained in G if and only if $\ell_x \cap \ell_y \neq \emptyset$. Hence G contains two families of 2–planes: the σ–planes (solid point–stars) and the ρ–planes (ruled planes) (cf. [SR, X, §4]). Let $h \subset \mathbb{P}^4$ be a hyperplane, let $p \in h$ be a point and let $w_2 \subset \mathbb{P}^4$ be a 2–plane. The σ–planes are the Schubert cycles $\sigma(p, h) = \{x \in G : p \in \ell_x \subset h\}$; the ρ–planes are the Schubert cycles $\sigma(w_2) = \{x \in G : \ell_x \subset w_2\}$.

Let $D(a_1, \ldots, a_k, n)$ be the flag variety of type $(a_1, \ldots, a_k, n)$, i.e., the variety that parametrizes flags of linear subspaces

$$V_{a_1} \subset V_{a_2} \subset \ldots \subset V_{a_k} \subset W,$$

where W is a complex vector space of dimension n and dim $V_i = i$. Instead of $D(a_1, \ldots, a_k, n)$ we sometimes write $D(a_1, \ldots, a_k, W)$.

The flag variety $D = D(a_1, \ldots, a_k, n)$ carries a sequence of universal subbundles

$$H_{a_1} \subset H_{a_2} \subset \ldots H_{a_k} \subset H_n = W \otimes_{\mathbb{C}} \mathcal{O}_D.$$

Let $H_{i,j} = H_i / H_j$ $(i > j)$ be the induced quotient bundles. The exact sequence $0 \to H_j \to H_i \to H_{i,j} \to 0$ is obtained by pulling back the tautological exact sequence on the Grassmann variety $G(a_j, a_i)$ via the projection map

$$p_{i,j} : D(a_1, \ldots, a_k, n) \to G(a_i, a_j).$$

The family of σ–planes on G is parametrized by the 7–dimensional flag variety $D = D(1, 4, 5)$; the family of ρ–planes on G is parametrized by the 6–dimensional flag variety $D(3, 5)$. In the sequel we shall concentrate on the family of σ–planes on G. The Plücker embedding $i : G(2, 5) \to \mathbb{P}^9$ sends a two–dimensional linear subspace $V_2 = \langle v_1, v_2\rangle$ to the line in $\bigwedge^2 V$ spanned by $v_1 \wedge v_2$. A coordinate–free description of the Plücker embedding is

$$\begin{array}{rcl} i : G(2,V) & \longrightarrow & \mathbb{P}(\bigwedge^2 V) \\ (V_2, V) & \mapsto & (\bigwedge^2 V_2, \bigwedge^2 V). \end{array}$$

Note that i is an embedding because the pair $(W, \bigwedge^2 V) \in i(G)$ uniquely determines V_2 by

$$V_2 = \{v \in V : v \wedge w = 0 \text{ for all } w \in W\}.$$

Given a point $(V_1, V_4, V) \in D(1,4,5)$, we denote by $V_1 \bigwedge V_4$ the subspace of $\bigwedge^2 V$ spanned by the vectors $v \wedge w$, where $v \in V_1$ and $w \in V_4$. The Plücker embedding induces an embedding of the flag variety $D(1,4,5)$ into the Grassmann variety $G' = G(3,10)$ of 2–planes in $\mathbb{P}^9$: choose a vector v that spans V_1 and a basis $\{v, v_1, v_2, v_3\}$ for V_4, and map the point (V_1, V_4, V) to the 3–dimensional linear subspace of $\bigwedge^2 V$ spanned by $\{v \wedge v_1, v \wedge v_2, v \wedge v_3\}$. A coordinate–free description of this map is

$$\begin{array}{rcl} j : D = D(1,4,5) & \rightarrow & G(3,10) \\ (V_1, V_4, V) & \mapsto & (V_1 \bigwedge V_4, \bigwedge^2 V). \end{array}$$

Note that we can recover the pair (V_1, V_4) from $(W, \bigwedge^2 V) \in \operatorname{im} j$ by setting

$$\begin{array}{c} V_1 = \{v \in V : v \wedge w = 0 \text{ for all } w \in W\} \\ V_4 = \{v \in V : v \wedge w = 0 \text{ for some } w \in W\}. \end{array}$$

Let $X = G \cap Q$ be a quadratic line complex. The quadric Q corresponds to a symmetric form $Q \in S^2(\bigwedge^2 V^\vee)$. Let

$$0 \rightarrow \mathcal{S}_3 \rightarrow \bigwedge^2 V \otimes \mathcal{O}_{G'} \rightarrow \mathcal{Q}_7 \rightarrow 0$$

be the tautological exact sequence on $G' = G(3,10)$. This sequence induces a surjective map of vector bundles

$$S^2(\bigwedge^2 V^\vee) \otimes \mathcal{O}_{G'} \rightarrow S^2 \mathcal{S}_3^\vee$$

whose kernel we denote by K. Let $s : S^2(\bigwedge^2 V^\vee) \rightarrow H^0(G', S^2\mathcal{S}_3^\vee)$ be the induced map on global sections. The Fano variety F_X of σ–planes contained in X is the zero scheme of the section $s(Q)$. Let

$$0 \rightarrow j^*K \rightarrow S^2(\bigwedge^2 V^\vee) \otimes \mathcal{O}_D \rightarrow j^* S^2 \mathcal{S}_3^\vee \rightarrow 0$$

be the exact sequence obtained by pullback to D. By composition of the inclusion map $\mathbb{P}(j^*K) \subset \mathbb{P}(S^2 \bigwedge^2 V^\vee) \times D$ and projection onto the first factor, we obtain a map

$$\mathbb{P}(j^*K) \rightarrow \mathbb{P}(S^2 \bigwedge^2 V^\vee)$$

that exhibits the projective bundle $\mathbb{P}(j^*K)$ as the universal family of Fano schemes of σ–planes over the family of quadratic line complexes (cf. [AK]).

To calculate the numerical invariants of the Fano scheme F_X, we determine the Chern classes of $j^*\mathcal{S}_3^\vee$.

Lemma 5.2.1. *$j^*\mathcal{S}_3 = H_1 \otimes H_{4,1}$.*

Proof: The fiber of $j^*\mathcal{S}_3$ over a point $x = (V_1, V_4, V)$ is $V_1 \bigwedge V_4$. Since the natural map $V_1 \bigwedge V_4 \to V_1 \otimes (V_4/V_1)$ is a canonical isomorphism, we obtain the desired isomorphism of vector bundles. □

Remark 5.2.2. Let $G(d, n)$ be a Grassmann variety that is embedded in projective space by the Plücker embedding. Let $X = V(d_1, \ldots, d_r) \cap G$ be a complete intersection in G. The previous result, whose original proof was simplified by a suggestion of L. Manivel, gives a method to compute the numerical invariants of the Fano schemes $F_k(X)$ of k–planes contained in X. It simplifies the method of computation used in [Mar].

The flag variety $D = D(1, 4, 5)$ is the incidence correspondence in $\mathbb{P}^4 \times (\mathbb{P}^4)^\vee$ with projections $p : D \to G(4, 5) = (\mathbb{P}^4)^\vee$ and $q : D \to \mathbb{P}^4$. Note that $j^*\mathcal{S}_3 = H_1 \otimes H_{4,1} = q^*(\mathcal{Q}_{\mathbb{P}^4}(-1))$. To describe the Chow ring $\mathrm{CH}^*(D)$, we note that the projection p gives D the structure of a projective bundle $\mathbb{P}(\mathcal{S}_4)$ over $G(4, 5)$. Set $x = c_1(\mathcal{O}_D(1))$ and $h = c_1(\mathcal{S}_4^\vee)$. The Chow ring of D is

$$CH^*(D) \cong \mathbb{Z}[x, h]/(x^4 - hx^3 + h^2x^2 - h^3x + h^4, h^5).$$

The first Chern classes of the universal bundles $H_1 = q^*\mathcal{O}_{\mathbb{P}^4}(-1)$ and $H_4 = p^*\mathcal{S}_4$ are $c_1(H_1) = -x$, $c_1(H_4) = -h$. Using the exact sequence

$$0 \to H_{4,1}^\vee \to H_4^\vee \to H_1^\vee \to 0$$

we compute the Chern polynomial of $H_{4,1}^\vee$:

$$\begin{aligned} c(H_{4,1}^\vee) &= (1 + ht + h^2t^2 + h^3t^3 + h^4t^4)(1 + xt)^{-1} \\ &= 1 + (h - x)t + (h^2 - hx + x^2)t^2 + (h^3 - h^2x + hx^2 - x^3)t^3. \end{aligned}$$

Using Lemma 5.2.1, we find that the Chern classes of $j^*\mathcal{S}_3^\vee$ are

$$\begin{aligned} c_1(j^*\mathcal{S}_3^\vee) &= 3c_1(H_1^\vee) + c_1(H_{4,1}^\vee) = h + 2x \\ c_2(j^*\mathcal{S}_3^\vee) &= 3c_1(H_1^\vee)^2 + 2c_1(H_1^\vee)c_1(H_{4,1}^\vee) + c_2(H_{4,1}^\vee) \\ &= 2x^2 + hx + h^2 \\ c_3(j^*\mathcal{S}_3^\vee) &= c_1(H_1^\vee)^3 + c_1(H_1^\vee)^2c_1(H_{4,1}^\vee) + c_1(H_1^\vee)c_2(H_{4,1}^\vee) + c_3(H_{4,1}^\vee) \\ &= hx^2 + h^3. \end{aligned}$$

The top Chern class of $E = S^2(j^*\mathcal{S}_3^\vee)$ is

$$\begin{aligned} c_6(E) &= 8c_1(j^*\mathcal{S}_3^\vee)c_2(j^*\mathcal{S}_3^\vee)c_3(j^*\mathcal{S}_3^\vee) - 8c_3(j^*\mathcal{S}_3^\vee)^2 \\ &= 32hx^5 + 24h^2x^4 + 56h^3x^3 + 24h^4x^2 + 24h^5x \\ &= 80h^3x^3. \end{aligned}$$

Let $\pi : \mathcal{X} \to \mathbb{P}H^0(\mathbb{P}^9, \mathcal{O}_{\mathbb{P}^9}(2))$ be the universal family of quadratic line complexes. Set $X_t = \pi^{-1}(t)$.

Lemma 5.2.3. *If $X \subset G$ is a general quadratic line complex, then F_X is a smooth curve of genus 161.*

Proof: Consider the universal family of Fano schemes

$$p : \mathbb{P}(j^*K) \to \mathbb{P}(S^2 \textstyle\bigwedge^2 V^\vee).$$

Note that $p^{-1}(t) = F_{X_t} = D \cap F_2(Q_t)$, where $F_2(Q_t)$ is the Fano variety of 2–planes contained in the quadric Q_t. For a general $Q \in \mathbb{P}(\bigwedge^2 S^2 V^\vee)$ we shall compute the intersection $[D].[F_2(Q)] \in \mathrm{CH}^{20}(G')$, $G' \cong G(3, 10)$. Because $[F_2(Q)] = c_6(S^2\mathcal{S}_3^\vee)$, we have

$$\begin{aligned} [D].[F_2(Q)] &= j_*(j^*[F_2(Q)]) \\ &= j_*c_6(E) \\ &= j_*(80h^3x^3). \end{aligned}$$

The projection formula shows that

$$\begin{aligned} j_*(80h^3x^3).c_1(\mathcal{S}_3^\vee) &= j_*(80h^3x^3.j^*c_1(\mathcal{S}_3^\vee)) \\ &= j_*(80h^3x^3.(2x + h)) \\ &= 240, \end{aligned}$$

where we have used that $h^3x^4 = h^4x^3$. Hence $[D].[F_2(Q)] = j_*(80h^3x^3) \neq 0$ and $D \cap F_2(Q) \neq \emptyset$ for general Q by Kleiman's transversality theorem [H2, III, Thm. 10.8]. It follows that the map p is dominant, and hence surjective. As $\mathbb{P}(j^*K)$ is a smooth and irreducible variety of dimension 55, the general fiber F_X is a smooth curve by generic smoothness [H2, III, Cor. 10.7]. The genus of F_X, for general X, is computed using the exact sequences

$$0 \to T_{F_X} \to T_D|_{F_X} \to E|_{F_X} \to 0 \tag{5.1}$$
$$0 \to T_v \to T_D \to p^*T_{G(4,5)} \to 0 \tag{5.2}$$
$$0 \to \mathcal{O}_D \to p^*\mathcal{S}_4 \otimes \mathcal{O}_D(1) \to T_v \to 0. \tag{5.3}$$

From the sequences (5.2) and (5.3) we obtain

$$c_1(T_D) = c_1(T_v) + c_1(p^*T_{G(4,5)}) = 4x - h + 5h = 4x + 4h.$$

Let $j_X : F_X \to D$ be the inclusion map. The exact sequence (5.1) shows that

$$\begin{aligned} (j_X)_* c_1(F_X) &= (c_1(T_D) - c_1(S^2\mathcal{S}_3^\vee)).[F_X] \\ &= (4x + 4h - 4(2x+h)).80h^3x^3 \\ &= -320h^3x^4 = -320h^4x^3, \end{aligned}$$

hence $2 - 2g(F_X) = -320$. □

To show that the curve F_X is connected, we study the homogeneous vector bundle $E^\vee$ and its exterior powers on the flag variety D. We refer to Section 4.2 for notation and terminology. Choose a basis $\{e_1, \ldots, e_5\}$ for V, and let $W \subset V$ be the subspace spanned by e_2, e_3 and e_4. Let $U \subset V$ be the one–dimensional subspace spanned by e_5. The flag variety D is a homogeneous space of the form $D = \mathrm{SL}(5,\mathbb{C})/P$, where

$$P = \left\{ \begin{pmatrix} h_1 & 0 & 0 \\ h_2 & h_3 & 0 \\ h_4 & h_5 & h_6 \end{pmatrix} : h_1,\ h_6 \in \mathbb{C}^*,\ h_3 \in \mathrm{GL}(3,\mathbb{C}), h_1.\det(h_3).h_6 = 1 \right\}.$$

Let $\rho : P \to W$ be the representation of P defined by $\rho(h) = h_3$, and let $\chi : P \to U$ be the character $\chi(h) = h_6$. The homogeneous vector bundle $j^*\mathcal{S}_3$ corresponds to the irreducible representation $\rho \otimes \chi : P \to W \otimes U$. Since

$$\begin{aligned} S^2(W \otimes U) &= S^2W \otimes U^{\otimes 2} \\ \bigwedge^m S^2(W \otimes U) &= \bigwedge^m (S^2W) \otimes U^{\otimes 2m}, \end{aligned}$$

it suffices to determine the highest weights of the representations $\bigwedge^m(S^2W)$ for $1 \le m \le 6$.

The representation ρ is induced by the standard representation of the semisimple part $P_{\mathrm{ss}} \cong \mathrm{SL}(3,\mathbb{C})$. The irreducible representation of $\mathrm{SL}(3,\mathbb{C})$ with highest weight $(\beta_2, \beta_3, \beta_4) = \beta_2 e_2 + \beta_3 e_3 + \beta_4 e_4$ is denoted by $\Gamma_{\beta_2,\beta_3,\beta_4}$.

Lemma 5.2.4. *The decompositions of the exterior powers $\bigwedge^k(S^2W)$ into irreducible representations of $\mathrm{SL}(3,\mathbb{C})$ are*

$$\begin{aligned} S^2W &\cong \Gamma_{2,0,0} & \bigwedge^4(S^2W) &\cong \Gamma_{4,3,1} \\ \bigwedge^2(S^2W) &\cong \Gamma_{3,1,0} & \bigwedge^5(S^2W) &\cong \Gamma_{4,4,2} \\ \bigwedge^3(S^2W) &\cong \Gamma_{4,1,1} \oplus \Gamma_{3,3,0} & \bigwedge^6(S^2W) &\cong \Gamma_{4,4,4}. \end{aligned}$$

Proof: This follows either from direct computation of the weights or by applying Formula (2.6) in [JPW]. □

Note that a weight $\lambda = (\beta_1, \ldots, \beta_5)$ of $\mathrm{SL}(5, \mathbb{C})$ is singular if and only if there exist indices $1 \leq i < j \leq 5$ such that $\beta_i = \beta_j$. The index of $\lambda = (\beta_1, \ldots, \beta_5)$ is

$$\begin{aligned} \mathrm{index}(\lambda) &= \#\{\alpha \in R^+ : (\lambda, \alpha) < 0\} \\ &= \#\{(i,j) : 1 \leq i < j \leq 5,\ \beta_i < \beta_j\}. \end{aligned}$$

Let $\delta = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 4e_1 + 3e_2 + 2e_3 + e_4$ be the sum of the fundamental dominant weights.

Using Lemma 5.2.4, we make a table of the highest weights λ_i associated to the vector bundles $\bigwedge^k E^\vee$ and the indices of $\lambda_i + \delta$ (if the weight is singular, we put a bar). Note that the highest weight of an irreducible representation of P is dominant for the semisimple part $P_{\mathrm{ss}} \cong \mathrm{SL}(3, \mathbb{C})$ of P. To emphasize this we write $\beta_1 e_1 + \ldots + \beta_5 e_5 = (\beta_1; \beta_2, \beta_3, \beta_4; \beta_5)$.

	λ_i	$\mathrm{index}(\lambda_i + \delta)$
$E^\vee$	(0;2,0,0;2)	-
$\bigwedge^2 E^\vee$	(0;3,1,0;4)	-
$\bigwedge^3 E^\vee$	(0;4,1,1;6)	4
	(0;3,3,0;6)	-
$\bigwedge^4 E^\vee$	(0;4,3,1;8)	6
$\bigwedge^5 E^\vee$	(0;4,4,2;10)	6
$\bigwedge^6 E^\vee$	(0;4,4,4;12)	7

Lemma 5.2.5. *If $X \subset G$ is a general quadratic line complex, the curve F_X is connected.*

Proof: In Lemma 5.2.3 we showed that F_X is a smooth curve. Since F_X is the zero locus of the global section $s(Q) \in H^0(D, E)$, we have a Koszul resolution

$$0 \to \bigwedge^6 E^\vee \to \cdots \to \bigwedge^2 E^\vee \to E^\vee \to \mathcal{O}_D \to \mathcal{O}_{F_X} \to 0$$

for $\mathcal{O}_{F_X}$. Hence $H^0(F_X, \mathcal{O}_{F_X}) \cong H^0(D, \mathcal{O}_D) = \mathbb{C}$ if $H^p(D, \bigwedge^p E^\vee) = 0$ for $1 \leq p \leq 6$. This follows from Theorem 4.2.2, as the weights λ_i associated to $\bigwedge^p E^\vee$ are either singular or have $\mathrm{index}(\lambda_i + \delta) \neq p$. $\square$

Remark 5.2.6. The quadratic complex of lines in $\mathbb{P}^4$ has been studied from a different point of view by B. Segre [Se]. He considers the Fano variety $F_3(G)$ of 3–planes on $G(2,5)$. Since every 3–plane contained in G is a Schubert cycle $\sigma(p)$ of lines through a point $p \in \mathbb{P}^4$, $F_3(G)$ is isomorphic to $\mathbb{P}^4$. A point $p \in \mathbb{P}^4$ is called singular (with respect to X) if the corresponding 3–plane $\sigma(p)$ is tangent to the quadric $Q \subset \mathbb{P}^9$ that defines X. For a general quadratic line complex X, Segre claims the following results:

1. The singular points are parametrized by a sextic hypersurface $\Sigma \subset \mathbb{P}^4$.
2. The points $p \in \mathbb{P}^4$ such that $\operatorname{rank}(Q|_{\sigma(p)}) \leq 2$ (i.e., the restriction of Q to $\sigma(p)$ is a union of two planes) are parametrized by a smooth curve $C \subset \Sigma$ of degree 40 and genus 81.

To rephrase these results in modern language, we consider the map

$$\begin{array}{rcl} g : \mathbb{P}^4 & \rightarrow & G(4, \bigwedge^2 V) \\ (V_1, V_5) & \mapsto & (V_1 \bigwedge V_5, \bigwedge^2 V) \end{array}$$

that embeds $F_3(G) \cong \mathbb{P}^4$ as a subvariety of the Grassmann variety $G' = G(4, 10)$ of 3–planes in $\mathbb{P}^9$. Set $F = g^* \mathcal{S}_4$, and let $\mathcal{Q}_{\mathbb{P}^4}$ be the universal quotient bundle on $\mathbb{P}^4$. As before, one shows that $F = H_1 \otimes H_{5,1} = \mathcal{Q}_{\mathbb{P}^4}(-1)$. Pull back the natural map $S^2(\bigwedge^2 V^\vee) \otimes \mathcal{O}_{G'} \rightarrow S^2(\mathcal{S}_4^\vee)$ to obtain a map $\tilde{s} : S^2(\bigwedge^2 V^\vee) \otimes \mathcal{O}_{\mathbb{P}^4} \rightarrow S^2 F^\vee$. The image $\tilde{s}(Q) \in S^2 F^\vee$ corresponds to a symmetric bundle map $f : F \rightarrow F^\vee$. Let

$$D_k(f) = \{p \in \mathbb{P}^4 : \operatorname{corank} f(p) \geq k\}$$

be the kth degeneracy locus of f. If $(p, h) \in F_X$, then $Q \cap \sigma(p)$ contains the 2–plane $\sigma(p, h)$. Hence p is a singular point, and we have a well–defined map $\tau : F_X \rightarrow C = D_2(f)$ that sends (p, h) to p; the map τ is a double covering, ramified over $D_3(f)$. It follows that if Q is general, the degeneracy loci $\Sigma = D_1(f)$ and $C = D_2(f)$ have the expected codimension; using the formulas in [HT], we find that $\deg \Sigma = 6$ and $\deg C = 40$. I did not verify that $D_3(f) = \emptyset$; if this locus is empty, then σ is an unramified covering and the Riemann–Hurwitz formula shows that the genus of C is 81, as claimed by B. Segre.

5.3 Infinitesimal Abel–Jacobi map

We study the infinitesimal Abel–Jacobi mapping associated to the family of σ–planes on a general quadratic line complex $X \subset G(2, 5)$. Once we have shown that this map is nontrivial, it follows that the Jacobian $J(F_X)$ of the parameter curve surjects onto the intermediate Jacobian $J^3(X)$ by a monodromy argument.

To find the normal bundle $N_{L_0,G}$ of a σ–plane L_0 inside the Grassmann variety $G = G(2, 5)$, we consider the restriction of the universal bundles over Grassmannians to certain Schubert cycles. Let $G(r+1, V)$ be the Grassmann variety of r–planes in $\mathbb{P}(V)$, where V is a complex vector space of dimension $n+1$. We write L_x for the r–plane corresponding to a point $x \in G(r+1, V)$.

Let $h \subset \mathbb{P}(V)$ be a hyperplane and $p \in h$ a point. Consider the following types of Schubert cycles:

$$\begin{aligned} Z_1 &= \sigma(h) = \{x \in G : L_x \subset h\} \cong G(r+1, n) \\ Z_2 &= \sigma(p) = \{x \in G : p \in L_x\} \cong G(r, n) \\ Z_3 &= \sigma(p, h) = \{x \in G : p \in L_x \subset h\} \cong G(r, n-1). \end{aligned}$$

Let

$$0 \to \mathcal{S} \to V \otimes \mathcal{O}_G \to \mathcal{Q} \to 0$$

be the tautological exact sequence on $G(r+1, n+1)$, and let $\mathcal{S}_i$ (resp. $\mathcal{Q}_i$) be the universal subbundle (resp. quotient bundle) on the Grassmann variety Z_i $(i = 1, 2, 3)$.

Lemma 5.3.1.

(i) $\mathcal{S}|_{Z_1} = \mathcal{S}_1$, $\mathcal{Q}|_{Z_1} \cong \mathcal{Q}_1 \oplus \mathcal{O}_{Z_1}$.

(ii) $\mathcal{S}|_{Z_2} \cong \mathcal{S}_2 \oplus \mathcal{O}_{Z_2}$, $\mathcal{Q}|_{Z_2} = \mathcal{Q}_2$.

(iii) $\mathcal{S}|_{Z_3} \cong \mathcal{S}_3 \oplus \mathcal{O}_{Z_3}$, $\mathcal{Q}|_{Z_3} \cong \mathcal{Q}_3 \oplus \mathcal{O}_{Z_3}$.

Proof: (i): Clearly $\mathcal{S}|_{Z_1} = \mathcal{S}_1$. To prove the second assertion, we write $h = \mathbb{P}(W)$ and consider the exact commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{S}_1 & \to & W \otimes \mathcal{O}_{Z_1} & \to & \mathcal{Q}_1 & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{S}|_{Z_1} & \to & V \otimes \mathcal{O}_{Z_1} & \to & \mathcal{Q}|_{Z_1} & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathcal{O}_{Z_1} & = & \mathcal{O}_{Z_1} & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

that is induced by the inclusion $W \subset V$. The short exact sequence

$$0 \to \mathcal{Q}_1 \to \mathcal{Q}|_{Z_1} \to \mathcal{O}_{Z_1} \to 0$$

splits, since $H^1(Z_1, \mathcal{Q}_1) = 0$ by the Bott vanishing theorem. The assertion (ii) follows from (i) by duality. Since Z_3 is a Schubert cycle of type Z_2 inside the Grassmann variety Z_1, (iii) follows by combining (i) and (ii). $\square$

Remark 5.3.2. This Lemma is taken from [Pap]. The converse statement is also true: if the restrictions of $\mathcal{S}$ and $\mathcal{Q}$ to a subvariety $Z \subset G(r+1, n+1)$ split as indicated, then Z is a Schubert cycle of type Z_1, Z_2 or Z_3 [loc. cit.].

Lemma 5.3.3. *The normal bundles of the Schubert cycles Z_1, Z_2 and Z_3 are*

(i) $N_{Z_1,G} = \mathcal{S}_1^\vee$, $N_{Z_2,G} = \mathcal{Q}_2$.

(ii) $N_{Z_3,Z_1} = \mathcal{Q}_3$, $N_{Z_3,Z_2} = \mathcal{S}_3^\vee$.

(iii) $N_{Z_3,G} \cong \mathcal{Q}_3 \oplus \mathcal{S}_3^\vee \oplus \mathcal{O}_{Z_3}$.

Proof: (i): The first assertion is clear, since Z_1 is the zero locus of a section $s \in H^0(G, S^\vee)$. The second assertion follows by duality. Clearly (ii) follows from (i). To prove (iii), we use Lemma 5.3.1 to determine the quotient of $(\mathcal{S}^\vee \otimes \mathcal{Q})|_{Z_3}$ by $\mathcal{S}_3^\vee \otimes \mathcal{Q}_3$. □

We return to the Grassmann variety $G = G(2,5)$. Let L_0 be a Schubert cycle of type Z_3, i.e., a σ–plane. Let $\mathcal{Q}_{L_0}$ be the universal quotient bundle on $L_0 \cong \mathbb{P}^2$. Lemma 5.3.3 shows that

$$N_{L_0,G} = \mathcal{Q}_{L_0} \bigoplus \mathcal{O}_{L_0}(1) \bigoplus \mathcal{O}_{L_0}. \tag{5.4}$$

Remark 5.3.4. Let $\mathcal{S}_{T_1}$ be the universal subbundle on a ρ–plane $T_1 \cong (\mathbb{P}^2)^\vee$. One can show in a similar way that the normal bundle of T_1 in G is $N_{T_1,G} = \oplus^2 \mathcal{S}_{T_1}^\vee$.

Let $X \subset G$ be a general quadratic line complex. In Lemmas 5.2.3 and 5.2.5 we saw that the family of σ–planes on X is parametrized by a smooth, irreducible curve F_X of genus 161. Let

$$\Phi_{F_X} : F_X \to J^3(X)$$

be the Abel–Jacobi mapping associated to this family of planes (note that it is only well–defined up to translation). By the universal property of the Jacobian $J(F_X)$ this map factorizes over a map

$$\Phi : J(F_X) \to J^3(X).$$

Let

$$\begin{array}{ccc} I & \xrightarrow{q} & X \\ \downarrow{\scriptstyle p} & & \\ F_X & & \end{array}$$

be the incidence correspondence. The induced map

$$q_{*}\circ p^{*} : H_1(F_X, \mathbb{Z}) \to H_5(X, \mathbb{Z})$$

is called the *cylinder homomorphism* associated to the family F_X. It sends a 1–chain $\gamma \subset F_X$ to the 5–chain $\cup_{x\in\gamma} L_x$ swept out on X by the planes L_x, $x \in \gamma$. Under Poincaré duality the cylinder homomorphism corresponds to a homomorphism

$$\psi_{\mathbb{Z}} : H^1(F_X, \mathbb{Z}) \to H^5(X, \mathbb{Z}).$$

Its complexification $\psi_{\mathbb{C}}$ is a morphism of Hodge structures of type $(2, 2)$ that induces a map

$$\psi : H^0(\Omega^1_{F_X})^{\vee} = H^{0,1}(F_X) \to H^{2,3}(X) = H^2(\Omega^3_X)^{\vee}.$$

Choose a point $0 \in F_X$ and let $L_0 \subset X$ be the corresponding σ–plane. The following result is due to Griffiths and Welters. Note that the adjunction formula shows that $\det(N_{L_0,X}) \cong \mathcal{O}_{L_0}$.

Lemma 5.3.5.

(i) *The transpose of the infinitesimal Abel–Jacobi mapping is the composition of the maps*

$$\begin{array}{rcl} H^2(X, \Omega^3_X) & \longrightarrow & H^2(L_0, \Omega^3_X|_{L_0}) \\ H^2(L_0, \Omega^3_X|_{L_0}) & \longrightarrow & H^2(L_0, K_{L_0} \otimes \bigwedge^2 N_{L_0,X}) \\ H^2(L_0, K_{L_0} \otimes \bigwedge^2 N_{L_0,X}) & \xrightarrow{\sim} & H^0(L_0, N_{L_0,X})^{\vee} \\ H^0(L_0, N_{L_0,X})^{\vee} & \xrightarrow{\sim} & T^{\vee}_{F_X,0} \end{array}$$

(ii) *The composed map*

$$\tau : H^2(X, \Omega^3_X) \to H^2(L_0, K_{L_0} \otimes \textstyle\bigwedge^2 N_{L_0,X})$$

fits into a commutative diagram

$$\begin{array}{ccc} H^1(X, T_X(-1)) & \xrightarrow{\beta} & H^1(L_0, N_{L_0,X}(-1)) \\ \downarrow & & \downarrow \\ H^2(X, \Omega^3_X) & \xrightarrow{\tau} & H^2(L_0, \bigwedge^2 N_{L_0,X}(-3)) \\ \Big\downarrow & & \Big\downarrow \alpha \\ H^2(X, \bigwedge^2 T_G \otimes \mathcal{O}_X(-3)) & \longrightarrow & H^2(L_0, \bigwedge^2 N_{L_0,G}(-3)) \\ \downarrow & & \downarrow \\ \cdots & \longrightarrow & H^2(L_0, N_{L_0,X}(-1)) \\ \downarrow & & \downarrow \\ \cdots & \longrightarrow & 0 \end{array}$$

with exact columns.

Proof: For (i), see [Gr1, Thm. 2.25]. The proof of the second assertion is analogous to the proof of Lemma 3.3.1. $\square$

Lemma 5.3.6.

(i) $\ker\alpha \neq 0$.

(ii) β *is surjective.*

Proof: (i): The Hilbert scheme Hilb_X^P that parametrizes 2–planes in X is the union of F_X and a finite number of points (corresponding to the ρ–planes contained in X). Hence the tangent space at 0 to F_X is isomorphic to $H^0(L_0, N_{L_0,X})$. As

$$h^2(L_0, \bigwedge^2 N_{L_0,X}(-3)) = h^0(L_0, N_{L_0,X}) = 1$$

by Serre duality, Lemma 5.3.5 shows that

$$\ker\alpha \neq 0 \iff H^2(L_0, \bigwedge^2 N_{L_0,G}(-3)) = H^2(L_0, N_{L_0,X}(-1)).$$

We shall show that both cohomology groups vanish. Lemma 5.3.3 shows that

$$\bigwedge^2 N^\vee_{L_0,G} \cong \bigwedge^2(\mathcal{Q}^\vee_{L_0} \oplus \mathcal{O}_{L_0}(-1) \oplus \mathcal{O}_{L_0}) \cong \bigoplus^2 \mathcal{O}_{L_0}(-1) \oplus \Omega^1_{L_0} \oplus \mathcal{Q}^\vee_{L_0},$$

hence

$$H^2(L_0, \bigwedge^2 N_{L_0,G}(-3)) \cong H^0(L_0, \bigwedge^2 N^\vee_{L_0,G})^\vee = 0.$$

To show that $H^2(L_0, N_{L_0,X}(-1) = 0$, we invoke the exact sequence

$$0 \to \mathcal{O}_{L_0}(-2) \to N^\vee_{L_0,G} \to N^\vee_{L_0,X} \to 0 \qquad (5.5)$$

and Lemma 5.3.3 to obtain

$$\begin{aligned} H^2(L_0, N_{L_0,X}(-1)) &= H^0(L_0, N^\vee_{L_0,X}(-2))^\vee \\ &= H^0(L_0, N^\vee_{L_0,G}(-2))^\vee \\ &= H^0(L_0, \mathcal{Q}^\vee_{L_0}(-2) \oplus \mathcal{O}_{L_0}(-3) \oplus \mathcal{O}_{L_0}(-2))^\vee \\ &= 0. \end{aligned}$$

For part (ii) we consider the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & T_{L_0} & = & T_{L_0} & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & T_X \otimes \mathcal{O}_{L_0} & \to & T_G \otimes \mathcal{O}_{L_0} & \to & N_{X,G} \otimes \mathcal{O}_{L_0} & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & N_{L_0,X} & \to & N_{L_0,G} & \to & N_{X,G} \otimes \mathcal{O}_{L_0} & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0. & & & & \end{array}$$

It induces a commutative diagram

$$\begin{array}{ccc}
H^0(X, \mathcal{O}_X(1)) & \xrightarrow{\gamma_1} & H^0(L_0, \mathcal{O}_{L_0}(1)) \\
\downarrow & & \downarrow \gamma_2 \\
H^1(X, T_X(-1)) & \xrightarrow{\beta} & H^1(L_0, N_{L_0,X}(-1)).
\end{array}$$

The map γ_1 is surjective, since X and L_0 are projectively normal in $\mathbb{P}^9$. As Lemma 5.3.3 shows that

$$H^1(L_0, N_{L_0,G}(-1)) = H^1(L_0, \mathcal{Q}_{L_0}(-1) \bigoplus \mathcal{O}_{L_0} \bigoplus \mathcal{O}_{L_0}(-1)) = 0,$$

the map γ_2 is also surjective; hence β is surjective. $\square$

Corollary 5.3.7. *The map Φ is nontrivial.*

Proof: This follows from Lemmas 5.3.5 (ii) and 5.3.6 by a diagram chase: start with a nonzero element $x \in \ker \alpha$ and lift it to an element

$$y \in H^1(L_0, N_{L_0,X}(-1)).$$

As β is surjective, we can lift y to an element $z \in H^1(X, T_X(-1))$. Its image $w \in H^2(X, \Omega_X^3)$ satisfies $\tau(w) = x \neq 0$. Thus the transpose of Φ_* is nontrivial, and the assertion follows. $\square$

Let $\{X_t\}_{t \in \mathbb{P}^1}$ be a Lefschetz pencil in $\mathbb{P}H^0(\mathbb{P}^9, \mathcal{O}_{\mathbb{P}^9}(2))$ with $X_0 = X$. Let

$$\begin{array}{ccc}
\mathcal{I} & \xrightarrow{q} & \mathcal{X} \\
\Big\downarrow p & & \\
\mathcal{F} & &
\end{array}$$

be the relative incindence correspondence. Let $U' \subset \mathbb{P}^1$ (resp. $U'' \subset \mathbb{P}^1$) be the subset over which $\mathcal{X}$ (resp. $\mathcal{F}$) is smooth. Set $U = U' \cap U''$.

Lemma 5.3.8. *The cylinder homomorphism*

$$\psi_{\mathbb{Z}} : H^1(F_X, \mathbb{Z}) \to H^5(X, \mathbb{Z})$$

is equivariant with respect to the action of $\pi_1(U, 0)$.

Proof: Let $\gamma \subset U$ be a closed loop based at 0. Since $\mathcal{I}$ is smooth over U, we can cover γ by a finite number of contractible open subsets $U_{\alpha(1)}, \ldots, U_{\alpha(k)}$

such that a vector field v_i on $U_{\alpha(i)}$ lifts to a C^∞ vector field θ_i on $\mathcal{I}|_{U_{\alpha(i)}}$. Let $\{\varphi_{i,t}\}$ be the family of diffeomorphisms associated to θ_i. The map

$$\begin{aligned} F_i : U_{\alpha(i)} \times I_{t(i)} &\to \mathcal{I}|_{U_{\alpha(i)}} \\ (t,x) &\mapsto \varphi_{i,t}(x) \end{aligned}$$

is a C^∞ trivialisation of $\mathcal{I}$ over $U_{\alpha(i)}$. We have compatible trivialisations

$$\begin{aligned} G_i : U_{\alpha(i)} \times X_{t(i)} &\to \mathcal{X}|_{U_{\alpha(i)}} \\ H_i : U_{\alpha(i)} \times F_{t(i)} &\to \mathcal{F}|_{U_{\alpha(i)}} \end{aligned}$$

of $\mathcal{X}$ and $\mathcal{F}$ over $U_{\alpha(i)}$ associated to $p_*\theta_i$ and $q_*\theta_i$. As the maps p_t and q_t are compatible with these trivialisations, it follows that they are compatible with the geometric monodromy. Hence p_0^*, $(q_0)_*$ and $\psi_{\mathbb{Z}}$ are equivariant with respect to the action of $\pi_1(U,0)$. $\square$

Theorem 5.3.9. *If $X \subset G$ is a general quadratic line complex, the map $\Phi : J(F_X) \to J^3(X)$ is surjective.*

Proof: Lemma 5.3.7 shows that the Abel–Jacobi map Φ is nontrivial. Since $\pi_1(U',0)$ acts transitively on $H^5(X,\mathbb{Q})$ (see [V4, Lecture 4] or [DK, Exposé XVIII, 6.6.2]), the surjectivity of ψ and Φ follows from Lemma 5.3.8, because the images of $\pi_1(U,0)$ and $\pi_1(U',0)$ in $\operatorname{Aut} H^5(X,\mathbb{Z})$ coincide. $\square$

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