Non–vanishing for Koszul cohomology of curves

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Abstract

We study the relationship between rank $p+2$ Koszul classes and rank 2 vector bundles on a smooth curve. We show that every rank $p+2$ Koszul class is obtained from a rank 2 vector bundle and give an explicit nonvanishing theorem for Koszul classes arising in this way.

1 Introduction

Let X be a smooth complex projective variety. The geometry of X is reflected in the behaviour of the Koszul cohomology groups $K_{p,q}(X,L)$ introduced by Green [4], more specifically the vanishing/nonvanishing of certain Koszul cohomology groups. The fundamental result in this direction is the nonvanishing theorem of Green–Lazarsfeld [5]. This theorem states that if a line bundle L admits a decomposition $L = L_1 \otimes L_2$ with $r_i = h^0(X, L_i) - 1 \geq 1$ $(i = 1, 2)$ then $K_{r_1+r_2-1,1}(X, L) \neq 0$. Voisin [8, (1.1)] has given a different proof of this result under the hypothesis that L_1 and L_2 are globally generated.

The aim of this note is to give a more geometric approach to this type of problems. The starting point is the following construction due to Voisin. Given a rank two vector bundle E on X with determinant L, Voisin [10, (2.22)] defined a homomorphism

$$\varphi : S^p H^0(X, E) \otimes \bigwedge^{p+2} H^0(X, E) \to \bigwedge^p H^0(X, L) \otimes H^0(X, L).$$

By [10, Lemma 5], this homomorphism produces elements of $K_{p,1}(X, L)$. If we take $E = L_1 \oplus L_2$, we get back the classes constructed by Green and Lazarsfeld. As one of the referees pointed out to us, Koh and Stillman [7] had generalised the Green–Lazarsfeld construction before from a different point of view.

Recall that the *rank* of a Koszul class $\gamma \in K_{p,1}(X, L)$ is the minimal dimension of a linear subspace $W \subset H^0(X, L)$ such that γ is represented by an element in $\bigwedge^p W \otimes H^0(X, L)$; cf. [6, Definition 2.2]. (Note that the subspace W is uniquely determined if $p \geq 2$.) By definition, the Koszul classes constructed in this paper are of rank $p+2$ if the vector bundle E is indecomposable.

Section 3 contains the main results of this paper. We first give a necessary and sufficient condition for nonvanishing of Koszul classes on smooth curves obtained from rank 2 vector bundles (Theorem 3.1). This result generalises the nonvanishing theorem of Green–Lazarsfeld in the case of curves. Our second main result, Theorem 3.4, states that every rank $p+2$ Koszul class on a smooth curve comes from a rank two vector bundle. This theorem is a generalisation of [6, Theorem 6.7].

2 Preliminaries

2.1 The method of Voisin

Let E be a rank two vector bundle on a smooth projective variety X defined over an algebraically closed field k of characteristic zero. Write $L = \det E$ and $V = H^0(X, L)$, and let

$$d : \bigwedge^2 H^0(X, E) \to V$$

be the determinant map. Given $t \in H^0(X, E)$, define a linear map

$$d_t : H^0(X, L) \to V$$

by $d_t(u) = d(t \wedge u)$, and choose a subspace $U \subset H^0(X, E)$ with $U \cap \ker(d_t) = 0$. Suppose that $\dim(U) = p + 2$ with $p \geq 1$, and put $W = d_t(U) \cong U$. The restriction of d to $\bigwedge^2 U$ defines a map $\bigwedge^2 U \to V$, which we can view as an element of

$$\bigwedge^2 U^\vee \otimes V \cong \bigwedge^p U \otimes V.$$

Let

$$\gamma \in \bigwedge^p W \otimes V \subset \bigwedge^p V \otimes V$$

be the image of this element under the map d_t.

Following Voisin [10, (2.22)], we prove that γ defines a Koszul class in $K_{p,1}(X, L)$. To this end, we make the previous construction explicit using coordinates. If we choose a basis $\{e_1, \ldots, e_{p+3}\}$ of $\langle t \rangle \oplus U \subset H^0(X, E)$ such that $e_1 = t$, we have

$$\begin{aligned} \gamma \;=\; & \sum_{i<j} (-1)^{i+j} d(t \wedge e_2) \wedge \ldots \wedge \widehat{d(t \wedge e_i)} \wedge \ldots \\ & \ldots \wedge \widehat{d(t \wedge e_j)} \wedge \ldots \wedge d(t \wedge e_{p+3}) \otimes d(e_i \wedge e_j). \end{aligned} \tag{1}$$

As in [10] one shows that the image of the γ by the Koszul differential

$$\delta : \bigwedge^p V \otimes H^0(X, L) \to \bigwedge^{p-1} V \otimes S^2 H^0(X, L)$$

equals

$$\begin{aligned} & \sum_{i<j<k} (-1)^{i+j+k} d(t \wedge e_2) \wedge \ldots \widehat{d(t \wedge e_i)} \ldots \widehat{d(t \wedge e_j)} \ldots \widehat{d(t \wedge e_k)} \ldots \wedge d(t \wedge e_{p+3}) \\ & \otimes \{ d(t \wedge e_i) d(e_j \wedge e_k) - d(t \wedge e_j) d(e_i \wedge e_k) + d(t \wedge e_k) d(e_i \wedge e_j) \}. \end{aligned} \tag{2}$$

Lemma 2.1 (Voisin) *Given four elements w_1, w_2, w_3, $w \in H^0(X, E)$ we have the relation*

$$d(w \wedge w_1) d(w_2 \wedge w_3) - d(w \wedge w_2) d(w_1 \wedge w_3) + d(w \wedge w_3) d(w_1 \wedge w_2) = 0$$

in $H^0(X, L^2)$.

Proof: See [10, Lemma 5]. □

The previous lemma shows that γ belongs to the kernel of the Koszul differential

$$\delta_X : \bigwedge^p V \otimes H^0(X, L) \to \bigwedge^{p-1} V \otimes H^0(X, L^2).$$

Hence γ defines a Koszul class $[\gamma] \in K_{p,1}(X, L, W) \subseteq K_{p,1}(X, L)$. Clearly the given class only depends on t and W; we write $[\gamma] = \gamma(W, t)$.

2.2 The method of Green–Lazarsfeld

Let L_1, L_2 be two line bundles on a smooth projective variety X such that $r_i = h^0(X, L_i) - 1 \geq 1$ $(i = 1, 2)$. Write $L_i = M_i + F_i$ with M_i the mobile part and F_i the fixed part. Let B be the divisorial part of $F_1 \cap F_2$. It is possible to choose $s_i \in H^0(X, L_i)$ such that $V(s_1, s_2) = B \cup Z$ with $\mathrm{codim}\,(Z) \geq 2$. Set $L = L_1 \otimes L_2$, and put $t = (s_1, s_2) \in H^0(X, L_1 \oplus L_2)$, $W = \mathrm{im}(d_t) \subset H^0(X, L(-B))$. By construction $h^0(X, \mathcal{O}_X(B)) = 1$, hence $\dim W = r_1 + r_2 + 1$. By the previous discussion, we obtain a Koszul class $\gamma(W, t) \in K_{r_1+r_2-1,1}(X, L)$. We call such classes *Green–Lazarsfeld classes.*

Definition 2.2 *Given a nonnegative integer $k \geq 0$, let $K_{k,1}(X, L)_{\mathrm{GL}} \subseteq K_{k,1}(X, L)$ be the subspace generated by Green–Lazarsfeld classes for all decompositions $L = L_1 \otimes L_2$ with $k = r_1 + r_2 - 1$, $(r_1 \geq 1,\ r_2 \geq 1)$.*

2.3 The method of Koh–Stillman

Voisin's method produces syzygies of rank $\leq p+2$. It is known that rank $p+1$ syzygies are Green–Lazarsfeld syzygies; see e.g. [6, Corollary 5.2]. Rank $p + 2$ syzygies can be obtained in the following way. Suppose that L is a globally generated line bundle on a projective variety X, and let $[\gamma] \in K_{p,1}(X, L)$ be a nonzero class represented by an element $\gamma \in \bigwedge^p W \otimes V$ with $\dim W = p + 2$. We view γ as an element in $\bigwedge^2 W^\vee \otimes V \cong \mathrm{Hom}(\bigwedge^2 W, V)$. Following [6, Proof of Theorem 6.1] we consider the map

$$\gamma' : \bigwedge^2(\mathbb{C} \oplus W) = W \oplus \bigwedge^2 W \to V$$

defined by taking the direct sum of γ and the inclusion $W \hookrightarrow V$. If we choose a generator e_1 for the first summand and a basis $\{e_2, \ldots, e_{p+3}\}$ for W, we obtain a skew–symmetric $(p + 3) \times (p + 3)$ matrix A by setting

$$a_{ij} = \gamma'(e_i \wedge e_j).$$

By construction, the inclusion $W \to V$ corresponds to the map $\gamma'(e_1 \wedge -)$. This allows us to identify a_{1j} and e_j, $2 \leq j \leq p + 3$. Let α be the image of γ under the Koszul differential

$$\delta : \bigwedge^p V \otimes V \to \bigwedge^{p-1} V \otimes S^2 V.$$

Writing this out, we obtain

$$\alpha = \sum_{i<j<k} (-1)^{i+j+k} a_{12} \wedge \ldots \widehat{a_{1,i}} \ldots \widehat{a_{1,j}} \ldots \widehat{a_{1,k}} \ldots \wedge a_{1,p+3} \otimes \mathrm{Pf}_{1ijk}(A). \qquad (3)$$

As the elements $\{a_{12}, \dots a_{1,p+3}\} = \{e_2, \dots, e_{p+3}\}$ are linearly independent, this expression is nonzero if and only if at least one of the Pfaffians $\mathrm{Pf}_{1ijk}(A)$ is nonzero. Furthermore, since α maps to zero in $\bigwedge^{p-1} V \otimes H^0(X, L^2)$ the Pfaffians $\mathrm{Pf}_{1ijk}(A)$ have to vanish on the image of X.

The preceding discussion shows that every rank $p+2$ syzygy arises from a skew–symmetric $(p+3) \times (p+3)$ matrix A such that

(i) the elements $\{a_{12}, \dots a_{1,p+3}\}$ are linearly independent;

(ii) there exists a nonzero Pfaffian $\mathrm{Pf}_{1ijk}(A)$;

(iii) the Pfaffians $\mathrm{Pf}_{1ijk}(A)$ vanish on the image of X in $\mathbb{P}(V^\vee)$.

This is exactly the method used by Koh and Stillman to produce syzygies; see [7, Lemma 1.3].

Remark 2.3 In the geometric setting of subsection 2.1, let Y be the image of X in $\mathbb{P}(V^\vee)$. The expression (2) shows that the canonical isomorphism

$$K_{p,1}(X, L) \cong K_{p-1,2}(\mathbb{P}^r, \mathcal{I}_Y, \mathcal{O}_{\mathbb{P}}(1))$$

maps the class $\gamma(W, t)$ to the element α defined in (3). Moreover, if d does not vanish on decomposable elements then $\gamma(W, t) \neq 0$. Indeed, this condition is satisfied if and only if the matrix A has no generalised zero; cf. [7, Definition (1.1)]. One then applies [loc. cit., Remark p. 122].

3 Main results

Theorem 3.1 *Let X be a smooth curve, let L be a base–point free line bundle on X and let $W \subset H^0(X, L)$ be a linear subspace. Put $B = \mathrm{Bs}(W)$, and let t be a section of $H^0(X, \mathcal{O}_X(B))$ vanishing on B. Consider an extension*

$$0 \to \mathcal{O}_X(B) \to E \to L(-B) \to 0 \tag{4}$$

such that

$$W \subset (\ker H^0(X, L(-B)) \xrightarrow{\delta} H^1(X, \mathcal{O}_X(B))).$$

Then the Koszul class $\gamma(W, t)$ defined in section 2.1 is nonzero is and only if the extension (4) is non-split.

Proof: The proof proceeds in several steps. We use the notation of section 2.1.

Step 1. Suppose that the extension (4) splits. In this case, one readily verifies that d vanishes identically on $\bigwedge^2 U$. The formula (1) then shows that $\gamma(W, t) = 0$.

Step 2. If $\gamma(W, t) = 0$ there exists a linear map $h : U \to \mathbb{C}$ such that

$$d(u_1 \wedge u_2) = h(u_2) d_t(u_1) - h(u_1) d_t(u_2) \tag{5}$$

for all $u_1, u_2 \in U$.

Indeed, suppose that there exists a nonzero element $\tilde{\gamma} \in \bigwedge^{p+1} W \cong W^\vee$ such that γ is the image of $\tilde{\gamma}$ under the Koszul differential. Then γ coincides with the composition of maps

$$\bigwedge^2 W \xrightarrow{\delta} W \otimes W \xrightarrow{\tilde{\gamma}\otimes \mathrm{id}} W \hookrightarrow V.$$

Since

$$\begin{aligned} d(u_1 \wedge u_2) &= \gamma(d_t(u_1) \wedge d_t(u_2)) \\ &= \tilde{\gamma}(d_t(u_2))d_t(u_1) - \tilde{\gamma}(d_t(u_1))d_t(u_2)), \end{aligned}$$

condition (5) is satisfied with $h = \tilde{\gamma}\circ d_t : U \to \mathbb{C}$.

Step 3. Let u_1, $u_2 \in U$ be two sections such that $d_t(u_1)$ and $d_t(u_2)$ generate $L(-B)$. If $d(u_1 \wedge u_2) = 0$, the extension (4) splits.

To prove this assertion, put $s_i = d_t(u_i)$ $(i = 1, 2)$ and consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}_X(B) & \to & E & \to & L(-B) & \to & 0 \\ & & & & \uparrow{\scriptstyle \mathrm{ev}_1} & & \uparrow{\scriptstyle \mathrm{ev}_2} & & \\ & & 0 & \to & \langle u_1 \,, u_2\rangle \otimes \mathcal{O}_X & \xrightarrow{\sim} & \langle s_1 \,, s_2\rangle \otimes \mathcal{O}_X & \to & 0. \end{array}$$

Put $M = \ker(\mathrm{ev}_1)$, and note that $\ker(\mathrm{ev}_2) \cong L^{-1}(B)$ since ev_2 is surjective. By the Snake Lemma we obtain an exact sequence

$$0 \to M \to L^{-1}(B) \to \mathcal{O}_X(B) \to \mathrm{coker}\,(\mathrm{ev}_1) \to 0.$$

Note that

$$d(u_1 \wedge u_2) = 0 \Longleftrightarrow \text{rank im}(\langle u_1 \,, u_2\rangle \otimes \mathcal{O}_X \to E) = 1 \Longleftrightarrow \text{rank } M = 1$$

where the first equivalence follows from [9, p. 380]. If $d(u_1 \wedge u_2) = 0$ the above exact sequence shows that $M \cong L^{-1}(B)$, hence the isomorphism $\langle u_1 \,, u_2\rangle \otimes \mathcal{O}_X \xrightarrow{\sim} \langle s_1 \,, s_2\rangle \otimes \mathcal{O}_X$ induces an isomorphism $\mathrm{im}(\mathrm{ev}_1) \cong L(-B)$. The inverse of this isomorphism provides a splitting of the extension (4).

Step 4. Suppose that $\gamma(W, t) = 0$. Then there exists a linear map $h : U \to \mathbb{C}$ as in Step 2. Consider the morphism

$$\pi : X \to \mathbb{P}(W^\vee)$$

defined by the base–point free linear system $W \subset H^0(X, L(-B))$, and choose a linear subspace $\Lambda \subset \mathbb{P}(W^\vee)$ of codimension two such that $\Lambda \cap \pi(X) = \emptyset$. The hyperplane $\ker(h) \subset W$ corresponds to a point $p \in \mathbb{P}(W^\vee)$. Put $H_1 = \langle \Lambda, p\rangle$ and choose a hyperplane $H_2 \subset \mathbb{P}(W^\vee)$ containing Λ such that $p \notin H_2$. Let u_1, u_2 be the sections corresponding to H_1, H_2. Then $d_t(u_1)$ and $d_t(u_2)$ generate $L(-B)$ and $u_1 \in \ker(h)$, $u_2 \notin \ker(h)$. Equation (5) yields the identity

$$d(u_1 \wedge u_2) = h(u_2)d_t(u_1).$$

Rewriting this identity, we obtain $d(u_1 \wedge (u_2 + h(u_2)t)) = 0$. Since the pair $\{d_t(u_1), d_t(u_2 + h(u_2)t)\} = \{d_t(u_1), d_t(u_2)\}$ generates $L(-B)$, Step 3 implies that the extension (4) splits. $\square$

Remark 3.2 In the statement of Theorem 3.1 it is not necessary to suppose that L is globally generated, since $K_{p,1}(X, L(-\operatorname{Bs}(L))) \cong K_{p,1}(X, L)$.

Theorem 3.1 yields a short, geometric proof of the Green–Lazarsfeld nonvanishing theorem for curves.

Theorem 3.3 (Green–Lazarsfeld) *Let X be a smooth curve, and let L be a line bundle on X that admits a decomposition $L = L_1 \otimes L_2$ with $r_i = \dim |L_i| \geq 1$ for $i = 1, 2$. Then $K_{r_1+r_2-1,1}(X, L) \neq 0$.*

Proof: We define s_1, s_2, t, W, B and $\gamma(W,t)$ as in section 2.2. Let C be the base locus of W, seen as a subspace of $H^0(X, L(-B))$. We prove that $\gamma(W,t) \neq 0$. Suppose that $\gamma(W,t) = 0$. Consider the extension

$$0 \to \mathcal{O}_X(B) \to L_1 \oplus L_2 \to L(-B) \to 0.$$

Pulling back this extension along the injective homomorphism $L(-B-C) \to L(-B)$, we obtain an induced extension

$$0 \to \mathcal{O}_X(B) \to E \to L(-B-C) \to 0.$$

Applying Theorem 3.1 to the line bundle $L(-C)$, we find that this extension splits. Hence there exists an injective homomorphism

$$\mathcal{O}_X(B) \oplus L(-B-C) \to L_1 \oplus L_2.$$

In particular there exists $i \in \{1, 2\}$ such that $\operatorname{Hom}(L(-B-C), L_i) \neq 0$. This implies that

$$r_i + 1 = h^0(X, L_i) \geq h^0(X, L(-B-C)) \geq \dim W = r_1 + r_2 + 1,$$

and this is impossible since $r_1 \geq 1$ and $r_2 \geq 1$. □

Theorem 3.4 *Let X be a smooth curve, and let $\alpha \neq 0 \in K_{p,1}(X, L)$ be a Koszul class of rank $p+2$ represented by an element of $\bigwedge^p W \otimes H^0(X, L)$ with* $\dim W = p+2$. *There exist a rank 2 vector bundle E on X and a section $t \in H^0(X, E)$ such that $\alpha = \gamma(W, t)$.*

Proof: Put $T = \mathbb{C} \oplus W$, and choose a basis $\{e_1, \dots, e_{p+3}\}$ of T such that $t = e_1$ is the generator of the first summand. Writing $z_{ij} = e_i \wedge e_j$, we obtain a skew–symmetric matrix $Z = (z_{ij})$ and coordinates $(z_{ij})_{1 \leq i < j \leq p+3}$ on $\mathbb{P}(\bigwedge^2 T^\vee)$. Consider the Grassmannian $G = G(2, T)$ of 2–dimensional quotients of T. The ideal of G under the Plücker embedding $G \subset \mathbb{P}(\bigwedge^2 T^\vee)$ is generated by the 4×4 Pfaffians $\operatorname{Pf}_{ijkl}(Z)$ of the matrix Z. Taking exterior powers in the exact sequence

$$0 \to \langle t \rangle \to T \to W \to 0$$

we obtain an exact sequence

$$0 \to \langle t \rangle \otimes W \to \bigwedge^2 T \to \bigwedge^2 W \to 0.$$

The linear subspace $\mathbb{P}(\bigwedge^2 W^\vee) \subset \mathbb{P}\left(\bigwedge^2 T^\vee\right)$ is defined by the vanishing of the linear forms z_{1j}, $j = 2, \ldots, p+3$. A straightforward computation then shows that the ideal of the union

$$G(2,T) \cup \mathbb{P}(\bigwedge^2 W^\vee) \subset \mathbb{P}(\bigwedge^2 T^\vee)$$

is generated by the Pfaffians $\mathrm{Pf}_{1ijk}(Z)$. The tautological exact sequence

$$0 \to S \to T \otimes \mathcal{O}_G \to Q \to 0$$

induces an isomorphism $T \cong H^0(G, Q)$. Under this isomorphism, we have $G(2, W) = V(t)$. As in section 2.3 we associate to the Koszul class α a matrix $A = (a_{ij})$ of linear forms $A = (a_{ij})$ such that

(a) The linear forms in the first row of A span W;

(b) There exists a nonzero 4×4 Pfaffian of A involving the first row and column;

(c) The 4×4 Pfaffians involving the first row and column of A vanish on the image of X in $\mathbb{P}H^0(X, L)^\vee$.

Let C be the base locus of the image of A. Replacing L by $L(-C)$ if necessary (W is obviously contained in the image of A) we can suppose that C is empty, hence the matrix A defines a morphism

$$\psi : X \to \mathbb{P}(\bigwedge^2 T^\vee).$$

Condition (c) implies that the image $Y = \psi(X)$ is contained in the union $G(2, T) \cup \mathbb{P}(\bigwedge^2 W^\vee)$, and condition (a) shows that Y is not contained in $\mathbb{P}(\bigwedge^2 W^\vee)$. As Y is irreducible, this implies that Y is contained in $G(2, T)$.

Put $E = \psi^* Q$. Twisting the exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{O}_G \to \psi_* \mathcal{O}_X \to 0$$

by the universal quotient bundle Q and taking global sections, we obtain an exact sequence

$$0 \to H^0(G, Q \otimes \mathcal{I}_Y) \to H^0(G, Q) \xrightarrow{\psi^*} H^0(G, \psi_* \mathcal{O}_X \otimes Q) \cong H^0(X, E).$$

Condition (a) implies that Y is not contained in $G(2, W) = G(2, T) \cap \mathbb{P}(\bigwedge^2 W^\vee)$, hence t does not vanish identically on X and defines a global section of E. The zero locus of this section is given by the equations $a_{12} = \ldots = a_{1,p+3} = 0$, hence it coincides with B. Consequently the line bundle E is given by an extension

$$0 \to \mathcal{O}_X(B) \to E \to L(-B) \to 0. \tag{6}$$

Consider the commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H^0(G, \mathcal{O}_G) & \xrightarrow{t} & H^0(X, \mathcal{O}_X(B)) \\
\downarrow{\scriptstyle \wedge t} & & \downarrow{\scriptstyle \wedge t} \\
H^0(G, Q) & \xrightarrow{\psi^*} & H^0(X, E) \\
\downarrow & & \downarrow{\scriptstyle d_t} \\
W & \xrightarrow{i} & H^0(X, L(-B)).
\end{array}$$

Note that $\ker i = W \cap H^0(G, \mathcal{O}_G(1) \otimes \mathcal{I}_Y) = 0$ by condition (a). As the map $H^0(G, Q) \to W$ is surjective, we find that W is contained in the image of the map $d_t : H^0(X, E) \to H^0(X, L(-B))$. Hence the condition of Theorem 3.1 is satisfied. By condition (b) we have $\gamma(W, t) \neq 0$. Hence the extension (6) does not split by Theorem 3.1. □

Remark 3.5 The union $G(2, T) \cup \mathbb{P}(\bigwedge^2 W^\vee)$ is a generic syzygy scheme; see [6, Theorem 6.1]. In [loc. cit., Theorem 6.7] it was shown that a rank $p+2$ syzygy gives rise to a rank 2 vector bundle if L is very ample and the ideal of X is generated by quadrics.

The condition of Theorem 3.1 can be reinterpreted in terms of surjectivity of a natural multiplication map.

Proposition 3.6 *Let X be a smooth curve, and let $W \subset H^0(X, L)$ be a linear subspace. We put $B = \mathrm{Bs}(W)$ and view W as a base–point free linear subspace of $H^0(X, L(-B))$. Let*

$$\mu : W \otimes H^0(X, K_X(-B)) \to H^0(K_X \otimes L(-2B))$$

be the multiplication map. The following conditions are equivalent.

(i) The map μ is not surjective;

(ii) There exists a non-split extension

$$0 \to \mathcal{O}_X(B) \to E \to L(-B) \to 0$$

such that W is contained in the kernel of the map $\delta : H^0(X, L(-B)) \to H^1(X, \mathcal{O}_X(B))$.

Proof: We first show that (i) implies (ii). Since μ is not surjective, there exists a hyperplane $H \subset H^0(X, K_X \otimes L(-B))$ that contains $\mathrm{im}(\mu)$. Let η be a linear functional defining H. Put $0 \neq \xi = \eta^\vee \in H^1(X, L^{-1}(B))$, and let

$$0 \to \mathcal{O}_X(B) \to E \to L(-B) \to 0$$

be the corresponding non-split extension. Given $w \in W$ and $v \in H^0(X, K_X(-B))$, the formula

$$\delta(w)(v) = (\eta \circ \mu)(w \otimes v) \tag{7}$$

shows that W is contained in the kernel of δ.

For the converse, note that formula (7) implies that $\eta|_{\mathrm{im}\,\mu} \equiv 0$. □

Remark 3.7 If B is a fixed divisor, the result of the previous Proposition follows from Green's duality theorem [4, Corollary (2.c.10)]. Indeed,

$$\text{coker } \mu \cong K_{0,1}(X, K_X(-B), L(-B), W) \cong K_{p,1}(X, B, L(-B), W)^\vee \tag{8}$$

and since $h^0(X, \mathcal{O}_X(B)) = 1$ we have an injection

$$K_{p,1}(X, B, L(-B), W) \hookrightarrow K_{p,1}(X, L).$$

Theorem 3.4 shows that Voisin's method may produce nontrivial Koszul classes that are not contained in the space $K_{p,1}(X, L)_{\mathrm{GL}}$ spanned by Green–Lazarsfeld classes.

Example 3.8 By [2, Theorem 3.6 and Theorem 4.3] there exists a smooth curve of genus 14 and Clifford index 5 whose Clifford index is computed by a unique line bundle L such that $L^2 = K_X$. The line bundle L embeds X in $\mathbb{P}^4$ as a projectively normal curve of degree 13 which is not contained in any quadric of rank ≤ 4, and the ideal of X is generated by the 4×4 Pfaffians of a skew–symmetric matrix $(a_{ij})_{1 \leq i,j \leq 5}$ with

$$\deg(a_{ij}) = \begin{cases} 2 \text{ if } i = 1 \text{ or } j = 1 \\ 1 \text{ if } i \geq 2 \text{ and } j \geq 2 \end{cases}$$

such that the quadric $Q = a_{23}a_{45} - a_{24}a_{35} + a_{25}a_{34}$ has maximal rank.

For any $p \geq 1$ there are no non-trivial Green-Lazarsfeld classes in $K_{p,1}(X, L)$. Indeed, suppose that $L = L_1 \otimes L_2$ with $h^0(X, L_i) \geq 2$ for $i \in \{1, 2\}$. Then $\deg(L_1) + \deg(L_2) = \deg(L) = 13$, hence either $\deg(L_1) \leq 6$ or $\deg(L_2) \leq 6$. But this implies that there exists $i \in \{1, 2\}$ such that $\mathrm{Cliff}(L_i) \leq 4$, contradiction.

The Koszul class $[Q] \in K_{1,1}(X, L)$ has rank 3, since it is represented by the linear subspace $W = \langle a_{23}, a_{24}, a_{25} \rangle$. Hence $[Q]$ comes from Voisin's method by Theorem 3.4.

Remark 3.9 A more geometric description of a subspace W representing $[Q]$ is the following. A smooth intersection of the quadric $V(Q) \subset \mathbb{P}H^0(X, L)^\vee$ with one of the cubic Pfaffians is a $K3$ surface in $\mathbb{P}H^0(X, L)^\vee$ containing a line ℓ which is disjoint from X by [2, Prop. 4.1]. The line ℓ corresponds to a 3-dimensional linear subspace $W \subset H^0(X, L)$, which is base-point-free since ℓ does not meet X.

One could ask whether the syzygies constructed in section 2.1 span $K_{p,1}(X, L)$. In principle it may be possible to obtain higher rank syzygies as linear combinations of rank $p + 2$ syzygies. However, if $K_{p,1}(X, L)$ is spanned by a single syzygy of rank $\geq p + 3$ this is not possible.

Example 3.10 (Eusen–Schreyer) Eusen and Schreyer [3, Theorem 1.7 (a)] have constructed a smooth curve $X \subset \mathbb{P}^5$ of genus 7 and Clifford index 3 embedded by the linear system $|K_X(-x)|$ such that $K_{2,1}(X, K_X(-x)) \cong \mathbb{C}$ is spanned by a syzygy s_0. The explicit expression for s_0 given on p.8 of [loc. cit.] shows that s_0 is a rank 5 syzygy. Hence s_0 cannot be obtained by the Green–Lazarsfeld construction or the method of section 2.1.

Acknowledgements. The first named author was partially supported by a Humboldt Research Fellowship, and by the ANCS grant 2-CEx 06-11-20/2006. We would like to thank Université Grenoble 1, I.H.E.S., Universität Bayreuth and Université Lille 1 for hospitality during the first stage of this work.

We thank the referees for several comments that helped to improve the presentation of the paper, and for pointing out an error in the previous version of this paper.

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