The regulator map for complete intersections

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To Professor Murre, with great respect.

1 Introduction

Since the introduction of the theory of infinitesimal variations of Hodge structure, infinitesimal methods have been successfully applied to a number of problems concerning the relationship between algebraic cycles and Hodge theory. One of the common techniques is to study infinitesimal invariants associated to families of algebraic cycles. This approach led to a proof of the infinitesimal Noether–Lefschetz theorem and was further developed by Green and Voisin in their study of the image of the Abel–Jacobi map for hypersurfaces in projective space. This work was reinterpreted and extended by Nori [18]. He proved a connectivity theorem for the universal family X_T of complete intersections of multidegree $(d_0, \ldots, d_r)$ on a polarised variety $(Y, \mathcal{O}_Y(1))$ inside the trivial family $Y_T = Y \times T$. Specifically, he proved that if the fibers of $X_T \to T$ are n–dimensional then $H^{n+k}(Y_T, X_T; \mathbb{Q}) = 0$ for all $k \leq n$ if $\min(d_0, \ldots, d_r)$ is sufficiently large. In [16] I proved an effective version of Nori's connectivity theorem; see [23] and [1] for related results in the case $Y = \mathbb{P}^N$.

For geometric applications of Nori's theorem one usually does not need the full strength of Nori's theorem; it often suffices to have $H^{n+k}(Y_T, X_T) = 0$ for all $k \leq c$, for some integer $c \leq n$. The theorems of Noether–Lefschetz and Green–Voisin can be deduced from Nori's theorem by taking $Y = \mathbb{P}^N$ and $(c, n) = (1, 2m)$ or $(c, n) = (2, 2m - 1)$. It turns out that in these cases, the degree bounds are sharp.

One can also use Nori's theorem to study the regulator maps on Bloch's higher Chow groups, as was noted in [21]. In [23] Voisin considered the extreme case $c = n$ of Nori's theorem for hypersurfaces in projective space and showed that also in this case the bound is sharp, by constructing interesting higher Chow cycles on hypersurfaces of low degree. One could therefore ask whether the degree bounds computed in [16, Thm. 3.13] are optimal for complete intersections in projective space. In this note we show that this is not the case, by studying the image of the regulator maps defined on the higher Chow groups $\mathrm{CH}^p(X,1)$ and $\mathrm{CH}^p(X,2)$. We improve the bounds computed in [16, Theorems. 4.4 and 4.6] using two methods: (1) a version of the Jacobi ring introduced in [1]; (2) a correspondence between the cohomology of complete intersections of quadrics and double coverings of projective space, combined with a version of Nori's theorem for cyclic coverings. These results are treated in sections 2 and 3. We conclude with an improved result on the image of the regulator map (Theorem 4.2) which is optimal for $\mathrm{CH}^p(X,2)$.

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2 Infinitesimal calculations

For the definition and basic properties of Bloch's higher Chow groups $\mathrm{CH}^p(X,q)$ we refer to [15]. There exist regulator maps

$$c_{p,q} : \mathrm{CH}^p(X,q) \to H^{2p-q}_{\mathcal{D}}(X,\mathbb{Z}(p))$$

that generalise the classical Deligne cycle class map; see [10] for an explicit description of these maps using integration currents. The starting point is the following result; cf. [16, Lemma 4.1] and the references cited there.

Proposition 2.1 *Let $U \subset \prod_{i=0}^{r} \mathbb{P}H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}}(d_i))$ be the open subset parametrising smooth complete intersections of dimension n and multidegree $(d_0,\ldots,d_r)$ in $\mathbb{P}^N$, and let $X_U \subset \mathbb{P}^N_U = \mathbb{P}^N \times U$ be the universal family. If*

$$H^k_{\mathcal{D}}(\mathbb{P}^N \times T, X_T) = 0$$

for all $k \leq 2p-q+1$ and for every smooth morphism $T \to U$, then the image of the regulator map

$$c_{p,q} : \mathrm{CH}^p(X,q) \otimes \mathbb{Q} \to H^{2p-k}_{\mathcal{D}}(X,\mathbb{Q}(p))$$

is contained in the image of the restriction map $H^{2p-q}_{\mathcal{D}}(\mathbb{P}^N, \mathbb{Q}(p)) \to H^{2p-q}_{\mathcal{D}}(X, \mathbb{Q}(p))$ *if* X *is very general.*

Using an effective version of Nori's connectivity theorem, we computed degree bounds for the cases $q = 1$ and $q = 2$.

Theorem 2.2 *Put*

$$\delta_{\min} = \min(d_0, \dots, d_r),\ \ \delta_{\max} = \max(d_0, \dots, d_r).$$

If $(n, q) \in \{(2m, 1), (2m-1, 2)\}$*, the conclusion of Proposition 2.1 holds if the following conditions are satisfied.*

(C_0) $\sum_{i=0}^r d_i + (m-1)\delta_{\min} \geq n + r + 3$*;*

(C_1) $\sum_{i=0}^r d_i + m\delta_{\min} \geq n + r + 2 + \delta_{\max}$*.*

Proof: See [16, Theorems 4.4 and 4.6]. □

Corollary 2.3 *If* $(n, q) = (2m, 1)$*, the conclusion of Proposition 2.1 holds, with the possible exception of the following cases.*

(i) $X = V(2) \subset \mathbb{P}^{2m+1}$*,* $X = V(3) \subset \mathbb{P}^3$*,* $X = V(4) \subset \mathbb{P}^3$*,* $X = V(3) \subset \mathbb{P}^5$*;*

(ii) $X = V(d, 2) \subset \mathbb{P}^{2m+2}$*,* $d \geq 2$*;*

(iii) $X = V(2, 2, 2) \subset \mathbb{P}^{2m+3}$*.*

Corollary 2.4 *If* $(n, q) = (2m-1, 2)$*, the conclusion of Proposition 2.1 holds, with the possible exception of the following cases.*

(i) $X = V(2) \subset \mathbb{P}^{2m}$*,* $X = V(3) \subset \mathbb{P}^2$*;*

(ii) $X = V(2, 2) \subset \mathbb{P}^{2m+1}$*,* $m \geq 1$*.*

Remark 2.5 Similar degree bounds can be worked out for $q \geq 3$. They coincide with the bounds of Corollary 2.3 (q odd) or Corollary 2.4 (q even). We have refrained from studying these cases as there is no description of $\mathrm{CH}^p(X, q)$ using Gersten–Quillen resolutions if $q \geq 3$.

To see whether the bounds of Corollaries 2.3 and 2.4 can be improved, we recall the idea of the proof of Theorem 2.2. Using mixed Hodge theory, one checks that it suffices to show

$$F^{m+1}H^{n+2}(\mathbb{P}^N_T, X_T) = 0 \tag{1}$$

in both cases. Let $f : X_T \to T$ be the structure morphism, and put $\mathcal{H}^{p,q}(X_T) = R^q f_* \Omega^p_{X_T/T}$. Let $\mathcal{H}^{p,q}_{\mathrm{pr}}(X_T)$ be the subbundle corresponding to primitive cohomology. By spectral sequence arguments one shows that the condition (1) is satisfied if the complex

$$0 \to \mathcal{H}^{p,n-p}_{\mathrm{pr}}(X_T) \to \Omega^1_T \otimes \mathcal{H}^{p-1,n-p+1}_{\mathrm{pr}}(X_T) \to \mathcal{H}^{p-2,n-p+2}_{\mathrm{pr}}(X_T) \tag{2}$$

is exact for all $p \geq m+1$.

Put $E = \bigoplus_{i=0}^r \mathcal{O}_{\mathbb{P}}(d_i)$, $P = \mathbb{P}(E^\vee)$ and set $\xi_E = \mathcal{O}_P(1)$. Let Σ be the sheaf of differential operators of order ≤ 1 on sections of ξ_E. Let X be a complete intersection defined by a section $s = (f_0, \ldots, f_r)$, and let σ be the corresponding section of $H^0(P, \xi_E)$. Contraction with the 1–jet $j^1(\sigma)$ defines maps $K_P \otimes \Sigma \otimes \xi_E^{a-1} \to K_P \otimes \xi_E^a$ for all $a \geq 1$. Put

$$\begin{aligned} J(K_P \otimes \xi_E^a) &= \mathrm{im}(H^0(P, K_P \otimes \Sigma \otimes \xi_E^{a-1}) \to H^0(P, K_P \otimes \xi_E^a)) \\ R(K_P \otimes \xi_E^a) &= H^0(P, K_P \otimes \xi_E^a)/J(K_P \otimes \xi_E^a). \end{aligned}$$

Proposition 2.6 *We have an isomorphism* $H^{n-p,p}_{\mathrm{pr}}(X) \cong R(K_P \otimes \xi_E^{p+r+1})$.

Proof: Cf. [5, Section 10.4] and the references cited there. □

The proof of Theorem 2.2 proceeds as follows. By semicontinuity it suffices to check the exactness of (2) pointwise. One can reduce to the case $T = H^0(\mathbb{P}^N, E) \setminus \Delta$, where Δ is the discriminant locus. Hence the tangent space to T is $V = H^0(\mathbb{P}^N, E)$. Applying these reductions to the dual of the complex (2), we see that it suffices to check the exactness of

$$\bigwedge^2 V \otimes H^{n-p+2,p-2}_{\mathrm{pr}}(X_t) \to V \otimes H^{n-p+1,p-1}_{\mathrm{pr}}(X_t) \to H^{n-p,p}_{\mathrm{pr}}(X_t) \to 0. \tag{3}$$

By Proposition 2.6, this complex is isomorphic to

$$R_\bullet = (\bigwedge^2 V \otimes R(K_P \otimes \xi_E^{p+r-1}) \to V \otimes R(K_P \otimes \xi_E^{p+r}) \to R(K_P \otimes \xi_E^{p+r+1}) \to 0).$$

Let $S_\bullet$ (resp. $J_\bullet$) be the complexes obtained by replacing the terms $R(K_P \otimes \xi_E^q)$ in $R_\bullet$ by $H^0(P, K_P \otimes \xi_E^q)$ (resp. $J(K_P \otimes \xi_E^q)$). The exact sequence of complexes $0 \to J_\bullet \to S_\bullet \to R_\bullet \to 0$ shows that $H_1(R_\bullet) = H_0(R_\bullet) = 0$ if

$$H_1(S_\bullet) = H_0(S_\bullet) = 0 \ (1), \quad H_0(J_\bullet) = 0 \ (2).$$

Using Castelnuovo–Mumford regularity, one then shows that (C_0) implies (1) and (C_1) implies (2).

It is possible to improve condition (ii) of Corollary 2.3. To this end, let $U' \subset \prod_{i=1}^r \mathbb{P}H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}}(d_i))$ be the open subset parametrising smooth complete intersections of multidegree $(d_1, \ldots, d_r)$, and let Y_U be the pullback of its universal family to $U \subset \prod_{i=0}^r \mathbb{P}H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}}(d_i))$. Given a smooth morphism $T \to U$, the inclusions of pairs

$$(Y_T, X_T) \subset (\mathbb{P}^N_T, X_T) \subset (\mathbb{P}^N_T, Y_T)$$

induce a long exact sequence

$$\to H^k(\mathbb{P}^N_T, Y_T) \to H^k(\mathbb{P}^N_T, X_T) \to H^k(Y_T, X_T) \to H^{k+1}(\mathbb{P}^N_T, X_T) \to$$

of cohomology groups. The vanishing of $H^k(Y_T, X_T)$ was investigated by Asakura and S. Saito [1]. For the vanishing of $H^{n+2}(Y_T, X_T)$ one introduces the bundles $\mathcal{H}^{p,q}(Y_T, X_T)$ of relative cohomology and studies exactness of the complex

$$0 \to \mathcal{H}^{p,n-p+1}(Y_T, X_T) \to \Omega^1_T \otimes \mathcal{H}^{p-1,n-p+2}(Y_T, X_T) \to \Omega^2_T \otimes \mathcal{H}^{p-2,n-p+3}(Y_T, X_T).$$

Exactness of this complex is reduced to exactness of

$$\bigwedge^2 V \otimes R'(K_P \otimes \xi_E^{p+r-1}) \to V \otimes R'(K_P \otimes \xi_E^{p+r}) \to R'(K_P \otimes \xi_E^{p+r+1}) \to 0 \quad (4)$$

where R' is the Jacobi ring defined in [1, §1].

Theorem 2.7 (Asakura–Saito) *Put* $d_{\max} = \max(d_1, \ldots, d_r)$. *The complex (4) is exact if*

(C_0) $\sum_{i=0}^r d_i + (m-1)\delta_{\min} \geq n + r + 3$;

(C_1') $\sum_{i=0}^r d_i + m\delta_{\min} \geq n + r + 2 + d_{\max}$.

Proof: See [1, Thm. 9-3 (ii)]. □

Corollary 2.8 *The conclusion of Proposition 2.1 holds if* $(n, q) = (2m, 1)$ *and* $(d_0, d_1) = (2, d)$ *if* $d \geq 4$.

Proof: If $(d_0, d_1) = (2, d)$ then Y_T is a family of odd–dimensional quadrics. As these quadrics have no primitive cohomology, a Leray spectral sequence argument shows that $H^k(\mathbb{P}^N_T, Y_T) = 0$ for all k. Hence we obtain isomorphisms $H^k(\mathbb{P}^N_T, X_T) \cong H^k(Y_T, X_T)$ for all k. Using Theorem 2.7 we obtain $H^k(\mathbb{P}^N_T, X_T) = 0$ for all $k \leq 2m+2$. Hence $H^k_{\mathcal{D}}(\mathbb{P}^N_T, X_T) = 0$ for all $k \leq 2m+2$. □

3 Complete intersections of quadrics

In this section we show how to exclude the exceptional cases of Corollary 2.3 (iii) and Corollary 2.4 (ii) using a correspondence between the cohomology of a double covering of projective space and the cohomology of a complete intersection of quadrics, and a version of Nori's theorem for double coverings (and more generally cyclic coverings) of projective space.

We start with Corollary 2.3 (iii). Let $X = V(Q_0, Q_1, Q_2) \subset \mathbb{P}^{2m+3}$ be a smooth complete intersection of three quadrics. Given $(\lambda_0 : \lambda_1 : \lambda_2) \in \mathbb{P}^2$, write $Q_\lambda = \lambda_0 Q_0 + \lambda_1 Q_1 + \lambda_2 Q_2$. (By abuse of notation, we use the same notation for a quadric Q, its defining equation and its associated symmetric matrix.) Let

$$\mathcal{X} = \{(x, \lambda) \in \mathbb{P}^{2m+3} \times \mathbb{P}^2 | x \in Q_\lambda\}$$

be the associated quadric bundle over $\mathbb{P}^2$, and let

$$C = \{\lambda \in \mathbb{P}^2 | \operatorname{corank}(Q_\lambda) \geq 1\}$$

be the discriminant curve.

The passage from the complete intersection X to the hypersurface $\mathcal{X} \subset \mathbb{P}^{2m+3} \times \mathbb{P}^2$ induces an isomorphism on middle dimensional primitive cohomology. This result is sometimes referred to as the Cayley trick; cf. [7, §6].

Proposition 3.1 (Cayley trick) *We have an isomorphism of Hodge structures* $H^{2m}_{\mathrm{pr}}(X)(-2) \cong H^{2m+4}_{\mathrm{pr}}(\mathcal{X})$.

The isomorphism of the previous Proposition is induced by the correspondence

$$\begin{array}{ccc} \Gamma_1 = f^{-1}(X) = X \times \mathbb{P}^2 & \rightarrow & \mathcal{X} \\ \downarrow f & & \\ X. & & \end{array}$$

As a smooth quadric in $\mathbb{P}^{2m+3}$ contains two families of $(m+1)$–planes, there exists a family Γ_2 of $(m+1)$–planes contained in the fibers of $f : \mathcal{X} \to \mathbb{P}^2$. The base of this family is a double covering $\pi : S \to \mathbb{P}^2$ that is ramified over the discriminant curve C.

Theorem 3.2 (O'Grady) *The correspondence* $\Gamma = \Gamma_2 \circ {}^t\Gamma_1$ *induces an isomorphism of Hodge structures* $H^2_{\mathrm{pr}}(S, \mathbb{Q}) \cong H^{2m}_{\mathrm{pr}}(X, \mathbb{Q})$*.*

Proof: See [19]; cf.[11] for a more general result, valid for arbitrary even–dimensional quadric bundles over $\mathbb{P}^2$.

□

The discriminant curve C is defined by the homogeneous polynomial

$$F(\lambda_0, \lambda_1, \lambda_2) = \det(\lambda_0 Q_0 + \lambda_1 Q_1 + \lambda_2 Q_2)$$

of degree $2m+4$; if the quadrics Q_0, Q_1 and Q_2 are general, C is smooth. Consider the vector bundle $E = \oplus^3 \mathcal{O}_{\mathbb{P}}(2)$ on $\mathbb{P}^{2m+3}$, and the map

$$H : H^0(\mathbb{P}^{2m+3}, E) \to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(2m+4))$$

that sends a net of quadrics to the equation of its discriminant curve. The map H induces a rational map

$$h : \mathbb{P}H^0(\mathbb{P}^{2m+3}, E) - - > \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(2m+4)).$$

Lemma 3.3 *Let* $U \subset \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(2m+4))$ *be the open subset parametrising smooth curves of degree* $2m+4$*. There exists a Zariski open subset* $U' \subset \mathbb{P}H^0(\mathbb{P}^{2m+3}, E) \setminus \Delta$ *such that* $h : U' \to U$ *is a smooth morphism.*

Proof: By a classical theorem of Dixon [8], a general smooth plane curve of even degree can be realised as the discriminant curve of a net of quadrics; see [2, Prop. 4.2 and Remark 4.4] for a modern proof. Hence the map h is dominant, and the assertion follows from [9, III, Lemma 10.5)]. □

In the sequel we shall need a relative version of Theorem 3.2. Over U we have the universal family $S_U \to U$ whose fiber over $[F] \in U$ is the double covering of $\mathbb{P}^2$ ramified over the curve $V(F)$. Let $S_T = S_U \times_U T$ be the pullback of this family to T along the map $h : T \to U$. Over T we have the family of quadric bundles $f_T : \mathcal{X}_T \to T$ associated to the family of complete intersections $X_T \to T$. We have relative correspondences

$$\begin{tikzcd} \Gamma_{1,T} \arrow[r] \arrow[d] & \mathcal{X}_T & \Gamma_{2,T} \arrow[r] \arrow[d] & \mathcal{X}_T \\ X_T & & S_T. & \end{tikzcd}$$

To state the relative version of Theorem 3.2 we need some notation. Given a map $f : Y \to X$ of topological spaces, let $M(f)$ be the mapping cylinder of f. The map f factors as $Y \hookrightarrow M(f) \xrightarrow{\sim} X$ where the second map is a homotopy equivalence. Define

$$H^k(X, Y; \mathbb{Z}) = H^k(M(f), \mathbb{Z}).$$

The groups $H^k(X, Y)$ fit into a long exact sequence

$$\to H^{k-1}(Y) \to H^k(X, Y) \to H^k(X) \xrightarrow{f^*} H^k(Y) \to \tag{5}$$

of cohomology groups. If f is the inclusion of a subspace, they coincide with the usual relative cohomology of the pair (X, Y).

Since we have isomorphisms

$$H^2_{\mathrm{pr}}(S) \cong H^3(\mathbb{P}^2, S), \quad H^{2m}_{\mathrm{pr}}(X) \cong H^{2m+1}(\mathbb{P}^{2m+3}, X),$$

Theorem 3.2 can be restated as an isomorphism $H^3(\mathbb{P}^2, S) \cong H^{2m+1}(\mathbb{P}^{2m+3}, X)$.

Theorem 3.4 *The relative correspondence $\Gamma_T = \Gamma_{2,T} \circ {}^t\Gamma_{1,T}$ induces an isomorphism $H^{k+2}(\mathbb{P}^2_T, S_T) \cong H^{2m+k}(\mathbb{P}^{2m+3}_T, X_T)$ for all $k \geq 0$.*

Proof: Let

$$f_T : X_T \to T, \; g_T : S_T \to T, \; \varphi_T : \mathbb{P}^{2m+3}_T \to T, \; \psi_T : \mathbb{P}^2_T \to T$$

be the projections onto the base T. By the Lefschetz hyperplane theorem and the Barth–Lefschetz theorem for cyclic coverings of projective space [12, Thm. 2.1] we have

$$\begin{aligned} R^q(f_T)_*\mathbb{Q} &\cong R^q(\varphi_T)_*\mathbb{Q}, \quad q \neq 2m \\ R^q(g_T)_*\mathbb{Q} &\cong R^q(\psi_T)_*\mathbb{Q}, \quad q \neq 2. \end{aligned}$$

Set $\mathbb{V} = \operatorname{coker}(R^{2m}(\varphi_T)_*\mathbb{Q} \to R^{2m}(f_T)_*\mathbb{Q})$, $\mathbb{W} = \operatorname{coker}(R^2(\psi_T)_*\mathbb{Q} \to R^2(g_T)_*\mathbb{Q})$. The correspondence Γ_T induces a homomorphism of local systems $\Gamma_{T,*} : \mathbb{W} \to \mathbb{V}$. By the proper base change theorem and Theorem 3.2, $\Gamma(t)_*$ is an isomorphism for all $t \in T$. Hence $\Gamma_{T,*} : \mathbb{W} \to \mathbb{V}$ is an isomorphism. Combining the long exact sequence (5) with the Lefschetz/Barth–Lefschetz isomorphisms we obtain

$$H^{2m+k}(\mathbb{P}^{2m+3}_T, X_T) \cong H^{k-1}(T, \mathbb{V}), \quad H^{k+2}(\mathbb{P}^2_T, S_T) \cong H^{k-1}(T, \mathbb{W}),$$

and the result follows. □

The vanishing of $H^*(\mathbb{P}^2_T, S_T)$ follows from an effective version of Nori's connectivity theorem for cyclic coverings of projective space, which can be seen as a generalisation of the result of Müller–Stach on the Abel–Jacobi map [13]. An outline of the proof can be found in [17].

Theorem 3.5 *Let $U \subset H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(d.e))$ be the open subset parametrising cyclic coverings $Y \to \mathbb{P}^n$ of degree e that ramify over a divisor $D \subset \mathbb{P}^n$ of degree $d.e$, and let $Y_U \to U$ be the universal family. Let $T \to U$ be a smooth morphism, and $c \leq n$ an integer. We have*

$$F^\mu H^{n+k}(\mathbb{P}^n_T, Y_T) = 0$$

for all $k \leq c$ if $(\mu - c)e + 1)d \geq n + c$.

Corollary 3.6 *The conclusion of Proposition 2.1 holds if $(n, q) = (2m, 1)$, $r = 2$ and $(d_0, d_1, d_2) = (2, 2, 2)$ if $m \geq 2$.*

Proof: Let U' be the open subset of $\mathbb{P}(\oplus^3 H^0(\mathbb{P}^{2m+3}, \mathcal{O}_{\mathbb{P}}(2))$ introduced in Lemma 3.3, and let $T \to U'$ be a smooth morphism. By Theorem 3.5 and Theorem 3.4 we obtain $H^{2m+1}(\mathbb{P}^{2m+3}_T, X_T) = H^{2m+2}(\mathbb{P}^{2m+3}_T, X_T) = 0$. □

In a similar way one can remove the exceptions in Corollary 2.4 (ii) for $m \geq 2$, using the following theorem of M. Reid [20].

Theorem 3.7 (Reid) *Let $X = V(2, 2) \subset \mathbb{P}^{2m+1}$ be a general smooth complete intersection of two quadrics. There is a family of m–planes in the fibers of the associated quadric bundle $\mathcal{X} \to \mathbb{P}^1$ that defines a correspondence Γ between $\mathcal{X}$ and a hyperelliptic curve C, branched over a divisor $D \subset \mathbb{P}^1$ of degree $2m + 2$. This correspondence induces an isomorphism*

$$\Gamma_* : H^1(C) \to H^{2m-1}(X).$$

4 Exceptional cases

We end with a discussion of the remaining exceptional cases. We start with the case $q = 1$. There are a number of trivial exceptions coming from the Noether–Lefschetz theorem. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^m(X) \otimes \mathbb{C}^* & \xrightarrow{\mu} & \mathrm{CH}^{m+1}(X,1) \\ \downarrow{\scriptstyle c_{m+1,1}} & & \downarrow{\scriptstyle c_{m+1,1}} \\ \mathrm{Hdg}^m(X) \otimes \mathbb{C}^* & \xrightarrow{\mu_{\mathcal{D}}} & H^{2m+1}_{\mathcal{D}}(X, \mathbb{Z}(m+1)). \end{array}$$

The composition of $\mu_{\mathcal{D}}$ with the projection

$$H^{2m+1}_{\mathcal{D}}(X, \mathbb{Z}(m+1)) = \frac{H^{2m+1}(X,\mathbb{C})}{F^{m+1}H^{2m+1}(X,\mathbb{C}) + H^{2m+1}(X,\mathbb{Z})} \to \frac{H^{m,m}(X)}{\mathrm{Hdg}^m(X)}$$

is an injective map

$$\mathrm{Hdg}^m(X) \otimes \mathbb{C}^* = \frac{\mathrm{Hdg}^m(X) \otimes \mathbb{C}}{\mathrm{Hdg}^m(X)} \hookrightarrow \frac{H^{m,m}(X)}{\mathrm{Hdg}^m(X)}.$$

Hence $\mu_{\mathcal{D}}$ is injective, and we obtain an injective map from $\mathrm{Hdg}^m_{\mathrm{pr}}(X)$ to the cokernel of $i^* : H^{2m+1}_{\mathcal{D}}(\mathbb{P}^N, \mathbb{Z}(m+1)) \to H^{2m+1}_{\mathcal{D}}(X, \mathbb{Z}(m+1))$. This remark covers the cases

$$X = V(2) \subset \mathbb{P}^{2m+1},\ X = V(3) \subset \mathbb{P}^3,\ X = V(2,2) \subset \mathbb{P}^{2m+2}.$$

The cycles that we considered above are *decomposable*, i.e., belong to the image of the map μ. The other counterexamples in low degree come from indecomposable higher Chow cycles. On K3 surfaces one can produce indecompable higher Chow cycles using rational nodal curves [4]; see [22], [14] and [6] for earlier results in this direction. Collino [6] gave examples of indecomposable higher Chow cycles on cubic fourfolds.

Remark 4.1 Note that both in the case of K3 surfaces and of cubic fourfolds we are dealing with a Hodge structure V of weight 2 with $\dim V^{2,0} = 1$. (In the case of a cubic fourfold X, take $V = H^4(X, \mathbb{C})(1)$.) Hence one might ask whether the existence of indecomposable higher Chow cycles on these varieties is related to the Kuga–Satake construction; cf. [22, 4.4-4.5].

The remaining exceptional case for $q = 1$ is $X = V(3, 2) \subset \mathbb{P}^{2m+2}$, $m \geq 2$. I do not know what happens in this case. With the notation of section 2, we can show that $H_0(S_\bullet) = 0$ and $H_1(S_\bullet) \neq 0$. (The latter result can be seen by decomposing the terms of the complex $S_\bullet$ into irreducible $\mathrm{SL}(V)$–modules.) Hence $H_1(R_\bullet) = 0$ if and only if the map $H_1(J_\bullet) \to H_1(S_\bullet)$ is surjective; it seems hard to verify this condition.

For $q = 2$ the situation is much simpler. The only cases to consider are

$$X = V(2) \subset \mathbb{P}^2,\ X = V(3) \subset \mathbb{P}^2, X = V(2, 2) \subset \mathbb{P}^3.$$

In the first case the conclusion of Proposition 2.1 trivially holds, since the target of the regulator map is zero. The remaining two cases are elliptic curves. Bloch [3] showed that the image of

$$c_{2,2} : \mathrm{CH}^2(X, 2) \to H^2_{\mathcal{D}}(X, \mathbb{Z}(2))$$

is nonzero for elliptic curves. Since $H^2_{\mathcal{D}}(\mathbb{P}^N, \mathbb{Z}(2)) = 0$, the result follows.

The results in this note can be summarised as follows.

Theorem 4.2 *Let X be a smooth complete intersection in $\mathbb{P}^N$ with inclusion map $i : X \to \mathbb{P}^N$.*

1) If $\dim X = 2m$ *and X is very general, the image of the regulator map*

$$c_{m+1,1} : \mathrm{CH}^{m+1}(X, 1) \to H^{2m+1}_{\mathcal{D}}(X, \mathbb{Z}(m + 1))/i^* H^{2m+1}_{\mathcal{D}}(\mathbb{P}^N, \mathbb{Z}(m + 1))$$

is a torsion group, with the exception of the cases

(i) $X = V(2) \subset \mathbb{P}^{2m+1}$, $X = V(d) \subset \mathbb{P}^3$ $(d \leq 4)$, $X = V(3) \subset \mathbb{P}^5$;

(ii) $X = V(2, 2) \subset \mathbb{P}^{2m+2}$, $X = V(3, 2) \subset \mathbb{P}^4$;

(iii) $X = V(2, 2, 2) \subset \mathbb{P}^5$

and the possible exception of the case $X = V(3, 2) \subset \mathbb{P}^{2m+2}$, $m \geq 2$.

2) If $\dim X = 2m - 1$ *and X is very general, the image of the map*

$$c_{m+1,2} : \mathrm{CH}^{m+1}(X, 2) \to H^{2m+1}_{\mathcal{D}}(X, \mathbb{Z}(m + 1))/i^* H^{2m+1}_{\mathcal{D}}(\mathbb{P}^N, \mathbb{Z}(m + 1))$$

is a torsion group, unless X is an elliptic curve.

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