A TORELLI TYPE THEOREM FOR THE MODULI SPACE OF RANK TWO CONNECTIONS ON A CURVE

INDRANIL BISWAS AND JAN NAGEL

Abstract. Let (X, x_0) be a pointed Riemann surface of genus $g \geq 3$, and let $\mathcal{M}_X$ be the moduli space parametrizing logarithmic $\mathrm{SL}(2,\mathbb{C})$–connections on X that are singular exactly over x_0 and have residue $-\mathrm{Id}/2$. We show that the moduli space $\mathcal{M}_X$ determines X up to isomorphism.

Résumé. Soit (X, x_0) une surface de Riemann pointée de genre $g \geq 3$, et soit $\mathcal{M}_X$ l'espace des modules des $\mathrm{SL}(2,\mathbb{C})$–connexions logarithmiques qui ont une singularité exactement en x_0 et ont pour résidu $-\mathrm{Id}/2$. On démontre que l'espace des modules $\mathcal{M}_X$ détermine X à isomorphisme près.

Version française abrégée. Soit X une courbe algébrique lisse sur $\mathbb{C}$ de genre $g \geq 3$. Dans cette note, on considère l'espace des modules $\mathcal{M}_X$ des couples (E, D), où E est un fibré vectoriel de rang 2 sur X et D est une connexion logarithmique sur E qui est singulière en exactement un point $x_0 \in X$ avec résidu $-\frac{\mathrm{Id}}{2}$. L'espace $\mathcal{M}_X$ ne dépend pas du choix de x_0. Le résultat principal est le théorème suivant.

Théorème. Soient X et Y des courbes lisses de genre $g \geq 3$. Si $\mathcal{M}_X \cong \mathcal{M}_Y$, alors $X \cong Y$.

L'idée de la démonstration est de se ramener au théorème de Torelli classique. On considère l'ouvert $\mathcal{M}_X^0$ de $\mathcal{M}_X$ des couples (E, D) tels que le fibré E est stable. Cet ouvert admet un morphisme vers l'espace des modules $\mathcal{N}_X$ des fibrés stables de rang 2 dont les fibres sont des espaces affines. Les espaces $\mathcal{M}_X^0$ et $\mathcal{N}_X$ ont donc la même cohomologie. On montre que le complémentaire $\mathcal{M}_X \setminus \mathcal{M}_X^0$ est de codimension ≥ 3. Ceci implique que $H^3(\mathcal{M}_X, \mathbb{Z}) \cong H^3(\mathcal{M}_X^0, \mathbb{Z})$. Si on combine les résultats précédents avec l'isomorphisme $H^3(\mathcal{N}_X, \mathbb{Z}) \cong H^1(X, Z)$ démontré par Mumford et Newstead [6], on obtient un isomorphisme de structures de Hodge $H^3(\mathcal{M}_X) \cong H^1(X)$ qui entraîne un isomorphisme entre la Jacobienne intermédiaire $J^2(\mathcal{M}_X)$ et la Jacobienne $J(X)$.

Pour terminer la démonstration, on montre qu'on peut retrouver la polarisation principale de $J^2(\mathcal{M}_X)$ à partir de $\mathcal{M}_X$. On considère la famille $r : \mathcal{M} \to M_g^0$, définie sur un ouvert de l'espace des modules des courbes de genre g, dont la fibre au–dessus de X est $\mathcal{M}_X$. Puis on construit un homomorphisme de systèmes locaux $\psi : \bigwedge^2 R^3 r_* \mathbb{Z} \to R^{6g-6} r_* \mathbb{Z}$. En utilisant le calcul des nombres de Betti de l'espace des modules des fibrés de Higgs de rang 2 par Hitchin [5], on montre que l'image de ψ est un système local de rang un. Avec un petit argument supplémentaire, on retrouve la polarisation principale à partir de la restriction $\psi|_{[X]}$.

1. Introduction

Let X be a compact connected Riemann surface of genus $g \geq 3$. Fix a point $x_0 \in X$. Let $\mathcal{M}_X$ denote the moduli space of all logarithmic connections (E, D) on X of the following type: $\text{rank}(E) = 2$ with $\bigwedge^2 E \cong \mathcal{O}_X(x_0)$ and D is a logarithmic connection on E singular exactly over x_0 with residue $-\frac{1}{2}\text{Id}_{E_{x_0}}$ such that the logarithmic connection on $\bigwedge^2 E$ induced by D coincides with the de Rham connection on $\mathcal{O}_X(x_0)$. The moduli space $\mathcal{M}_X$ is a smooth quasi-projective variety over $\mathbb{C}$. The isomorphism class of the variety $\mathcal{M}_X$ does not depend on the choice of the base point x_0.

We prove that the isomorphism class of the variety $\mathcal{M}_X$ determines X. In other words, if (Y, y_0) is another pointed Riemann surface and $\mathcal{M}_Y \cong \mathcal{M}_X$, then $X \cong Y$.

We note that the biholomorphism class of the complex manifold $\mathcal{M}_X$ is independent of the complex structure of X; the biholomorphism class depends only on g. Indeed, sending a logarithmic connection to its monodromy we get a holomorphic embedding

$$\rho : \mathcal{M}_X \longrightarrow \text{Hom}(\pi_1(X - x_0), \text{SL}(2, \mathbb{C}))/\text{SL}(2, \mathbb{C}).$$

The image of ρ is the complex submanifold $\mathcal{R} \subset \text{Hom}(\pi_1(X - x_0), \text{SL}(2, \mathbb{C}))/\text{SL}(2, \mathbb{C})$ consisting of homomorphisms that send the loop around x_0 to $-\text{Id}$. Therefore, the biholomorphism class of $\mathcal{M}_X$ is independent of the complex structure of X.

2. Moduli space of connections

Let X be a compact connected Riemann surface of genus g, with $g \geq 3$. The holomorphic cotangent bundle of X will be denoted by K_X. Fix a base point $x_0 \in X$.

Given a holomorphic vector bundle E over X, a *logarithmic connection* on E singular over x_0 is a first order holomorphic differential operator

$$D : E \longrightarrow E \otimes K_X \otimes \mathcal{O}_X(x_0) \tag{2.1}$$

which satisfies the Leibniz identity. Its *residue* $\text{Res}(D, x_0)$ is an element of $\text{End}(E_{x_0})$ [4].

Let $\mathcal{M}_X$ denote the moduli space of all pairs (E, D) of the following type:

(1) E is a rank two holomorphic vector bundle over X, with $\bigwedge^2 E = \mathcal{O}_X(x_0)$,
(2) D is a logarithmic connection on E singular over x_0, with residue $\text{Res}(D, x_0) = -\frac{1}{2}\text{Id}_{E_{x_0}}$, and
(3) the logarithmic connection on $\bigwedge^2 E$ induced by D coincides with the logarithmic connection on $\mathcal{O}_X(x_0)$ defined by the de Rham differential.

The moduli space $\mathcal{M}_X$ was constructed in [10] and [9] as an irreducible quasi-projective complex variety of dimension $6g - 6$. As any connection in $\mathcal{M}_X$ is irreducible [1, Lemma 2.3], the variety $\mathcal{M}_X$ is smooth.

Let

$$\mathcal{M}_X^0 \subset \mathcal{M}_X$$

be the Zariski open dense subset defined by all $(E\,,D) \in \mathcal{M}_X$ such that the underlying holomorphic vector bundle E is stable, and let

$$Z := \mathcal{M}_X \setminus \mathcal{M}_X^0 \subset \mathcal{M}_X \tag{2.2}$$

be the complement.

Proposition 2.1. *The (complex) codimension of the Zariski closed subset $Z \subset \mathcal{M}_X$ (see (2.2)) is g. In particular, the codimension of Z is at least three (recall that $g \geq 3$).*

Proof. Let E be an irreducible vector bundle E over X of rank two such that $\bigwedge^2 E \cong \mathcal{O}_X(x_0)$. Consider the space of all logarithmic connections D on E singular over x_0 such that $(E, D) \in \mathcal{M}_X$. This space, which we denote by $\mathcal{A}(E)$, is an affine space for $H^0(X,\, \mathrm{ad}(E) \otimes K_X)$, where $\mathrm{ad}(E) \subset \mathrm{End}(E)$ is the subbundle defined by trace zero endomorphisms.

The group $\mathcal{G}(E) := \mathrm{Aut}(E)/\mathbb{C}^*$ acts on the affine space $\mathcal{A}(E)$. Since any logarithmic connection in $\mathcal{A}(E)$ is irreducible by [1, Lemma 2.3], the action of $\mathcal{G}(E)$ on $\mathcal{A}(E)$ is faithful. Hence the dimension of the space of isomorphism classes of logarithmic connections on E is

$$\dim \mathcal{A}(E) - \dim \mathrm{Aut}(E) + 1 = h^1(X,\, \mathrm{ad}(E)) - h^0(X,\, \mathrm{End}(E)) + 1 = 3g - 3\,. \tag{2.3}$$

If E is not semistable, there exists a line subbundle $L \subset E$ of degree $d \geq 1$ giving an exact sequence

$$0 \longrightarrow L \longrightarrow E \longrightarrow L^{-1}(x_0) := L^{-1} \otimes_{\mathcal{O}_X} \mathcal{O}_X(x_0) \longrightarrow 0\,. \tag{2.4}$$

If we fix L, the space of all isomorphism classes of extensions of the above type is parametrized by $H^1(X,\, L^2(-x_0))/\mathbb{C}^*$. We have $\mathrm{Hom}(L^{-1}(x_0), L)) \subset \mathrm{ad}(E)$, and

$$\dim H^1(X,\, L^2(-x_0)) - \dim H^0(X,\, L^2(-x_0)) \;=\; g - 1 - d \;\leq\; g - 2\,.$$

Combining this with (2.3) we conclude that for fixed L, the dimension of the space of all $(E, D) \in \mathcal{M}_X$ such that E fits in an exact sequence as in (2.4) is at most $3g-3+g-3 = 4g-6$. Since $\dim J^d(X) = g$, this implies that the dimension of the subvariety Z in (2.2) is at most $4g-6+g = 5g-6$. Since $\dim(\mathcal{M}_X) = 6g-6$, the statement of the proposition follows. $\square$

Let $\mathcal{N}_X$ denote the moduli space of stable vector bundles E over X of rank two with $\bigwedge^2 E \cong \mathcal{O}_X(x_0)$. We have a natural projection

$$\phi : \mathcal{M}_X^0 \longrightarrow \mathcal{N}_X\,, \tag{2.5}$$

where $\mathcal{M}_X^0$ is as in (2.2), that sends a pair $(E\,, D)$ to E. Since $\mathcal{A}(E)$ (defined in the proof of Proposition 2.1) is an affine space for $H^0(X,\, \mathrm{ad}(E) \otimes K_X)$, it follows that the map ϕ in (2.5) makes $\mathcal{M}_X^0$ a $\Omega^1_{\mathcal{N}_X}$–torsor over $\mathcal{N}_X$.

Given a stable vector bundle $E \in \mathcal{N}_X$, there is a unique unitary flat connection D_E on $E_{X\setminus\{x_0\}}$ such that $(E\,, D_E) \in \mathcal{M}_X^0$ [8]. Consequently, we obtain a C^∞ section of the holomorphic fibration ϕ in (2.5)

$$\gamma : \mathcal{N}_X \longrightarrow \mathcal{M}_X^0 \subset \mathcal{M}_X \tag{2.6}$$

that sends E to $(E\,, D_E)$.

3. Intermediate Jacobian and polarization

The intermediate Jacobian associated to $H^3(\mathcal{M}_X)$ is defined by

$$J^2(\mathcal{M}_X) = \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-2)\,, H^3(\mathcal{M}_X)).$$

Carlson [3, Proposition 2] showed that

$$J^2(\mathcal{M}_X) \;\cong\; H^3(\mathcal{M}_X, \mathbb{C})/(F^2H^3(\mathcal{M}_X, \mathbb{C}) + H^3(\mathcal{M}_X, \mathbb{Z})).$$

A priori, $J^2(\mathcal{M}_X)$ is a generalized torus. In our case it is an abelian variety because of the following result.

Lemma 3.1. *We have an isomorphism of Hodge structures $H^3(\mathcal{M}_X, \mathbb{Z}) \;\cong\; H^1(X, \mathbb{Z})(-1)$.*

Proof. Consider the Zariski closed subset $Z \subset \mathcal{M}_X$ defined in (2.2). By Proposition 2.1 we have $\mathrm{codim}_{\mathcal{M}_X}(Z) \geq 3$. Hence $H^3_Z(\mathcal{M}_X) = H^4_Z(\mathcal{M}_X) = 0$ (weak purity) and

$$j^* \,:\, H^3(\mathcal{M}_X, \mathbb{Z}) \longrightarrow H^3(\mathcal{M}^0_X, \mathbb{Z})$$

is an isomorphism.

Since $\varphi : \mathcal{M}^0_X \to \mathcal{N}_X$ is an affine fiber bundle, the induced homomorphism $\varphi^* : H^3(\mathcal{N}_X, \mathbb{Z}) \longrightarrow H^3(\mathcal{M}^0_X, \mathbb{Z})$ is an isomorphism.

Mumford and Newstead showed that there exists an isomorphism

$$\Gamma_{X*} \,:\, H^1(X, \mathbb{Z})(-1) \longrightarrow H^3(\mathcal{N}_X, \mathbb{Z})$$

induced by a certain correspondence Γ_X between $\mathcal{N}_X$ and X; see [6, page 1201, Theorem]. The correspondence Γ_X in question is the second Chern class of the universal adjoint bundle over $X \times \mathcal{N}_X$. The isomorphism $(j^*)^{-1} \circ \varphi^* \circ \Gamma_{X*} : H^1(X, \mathbb{Z})(-1) \longrightarrow H^3(\mathcal{M}_X, \mathbb{Z})$ gives the required isomorphism in the lemma. ☐

Corollary 3.2. *We have an isomorphism $J^2(\mathcal{M}_X) \;\cong\; J(X)$. In particular, $J^2(\mathcal{M}_X)$ is an abelian variety.*

Our next aim is to recover the principal polarization on $J(X)$ from $\mathcal{M}_X$. To this end, we consider the universal curve

$$\mathcal{C} = M^0_{g,1} \xrightarrow{\;p\;} M^0_g$$

over the moduli space of curves of genus g without nontrivial automorphisms. Let

$$\mathcal{J} \xrightarrow{\;q\;} M^0_g, \quad \mathcal{M} \xrightarrow{\;r\;} M^0_g$$

be the families whose fibers over $[X] \in M^0_g$ are $q^{-1}(X) = J(X)$, $r^{-1}(X) = \mathcal{M}_X$.

Consider the local system $R^2q_*\mathbb{Z}$ on M_g^0. For any curve X, the Jacobian $J(X)$ has a canonical principal polarization which gives a constant sublocal system

$$\mathbb{L} \subset R^2q_*\mathbb{Z}\,.$$

Proposition 3.3. *The local system $R^2q_*\mathbb{Z}$ on M_g^0 decomposes as*

$$R^2q_*\mathbb{Z} \;=\; \mathbb{L} \oplus (R^2q_*\mathbb{Z})_0\,.$$

The local system $(R^2q_*\mathbb{Z})_0$ is irreducible.

Proof. Since $(R^2q_*\mathbb{Z})_0$ is torsionfree, it suffices to show that the local system of primitive cohomology $(R^2q_*\mathbb{C})_{\mathrm{prim}} \;=\; (R^2q_*\mathbb{Z})_0 \otimes_{\mathbb{Z}} \mathbb{C}$ is irreducible. If we fix a base point $X_0 \;\in\; M_g^0$, then the mapping class group (the monodromy for the family p) surjects onto $\mathrm{Aut}(H^1(X_0,\mathbb{Z})) \;\cong\; \mathrm{Sp}(2g,\mathbb{Z})$, and the Borel density theorem says that $\mathrm{Sp}(2g,\mathbb{Z})$ is Zariski dense in $\mathrm{Sp}(2g,\mathbb{C})$ [2]. Since the Lefschetz decomposition of $H^i(J(X_0),\mathbb{C})$ for the canonical principal polarization on $J(X_0)$ corresponds to the decomposition of the $\mathrm{Sp}(H^1(X_0,\mathbb{C}))$–module $H^i(J(X_0),\mathbb{C}) \;=\; \bigwedge^i H^1(X_0,\mathbb{C})$ into irreducible components, the proposition follows. $\square$

We have $H^2(\mathcal{M}_X,\mathbb{Z}) \cong H^2(\mathcal{N}_X,\mathbb{Z})$ (follows from Proposition 2.1), and $H^2(\mathcal{N}_X,\mathbb{Z}) \;=\; \mathbb{Z}$ [7, Theorem 3]. Therefore, $R^2r_*\mathbb{Z} \cong \mathbb{Z}$ (the ample generator of $H^2(\mathcal{N}_X,\mathbb{Z})$ gives a trivialization of $R^2r_*\mathbb{Z}$). Let η denote the section of $R^2r_*\mathbb{Z}$ given by the ample generator of $H^2(\mathcal{N}_X,\mathbb{Z})$.

Consider the homomorphism of local systems

$$\psi \;:\; \bigwedge\nolimits^2 R^3r_*\mathbb{Z} \;\longrightarrow\; R^{6g-6}r_*\mathbb{Z} \tag{3.1}$$

defined by $\psi(\alpha\wedge\beta) \;=\; \alpha\cup\beta\cup\eta^{3g-6}$.

Lemma 3.4. *The image of ψ is a rank one local system.*

Proof. The earlier mentioned result of Mumford and Newstead says that we have an isomorphism of local systems

$$\Gamma_* \;:\; R^1p_*\mathbb{Z} \;\longrightarrow\; R^3r_*\mathbb{Z} \tag{3.2}$$

defined by the correspondence Γ on the fiber product $M_{g,1}^0 \times_{M_g^0} \mathcal{M}$ given by the second Chern class of the universal adjoint bundle; note that $R^1p_*\mathbb{Z}$ is self–dual.

Define

$$\psi' = \psi\circ \bigwedge\nolimits^2\Gamma_* \;:\; \bigwedge\nolimits^2 R^1p_*\mathbb{Z} \;\longrightarrow\; R^{6g-6}r_*\mathbb{Z}\,.$$

We recall that the local system $(R^2q_*\mathbb{Z})_0$ is irreducible of rank $g(2g-1)-1$ (see Proposition 3.3). We also know that $R^2q_*\mathbb{Z} \cong \bigwedge^2 R^1p_*\mathbb{Z}$.

On the other hand, the rank of $R^{6g-6}r_*\mathbb{Z}$ is g [5, Theorem 7.6]. As an irreducible local system does not admit any nonzero homomorphism to a local system of lower rank, the homomorphism ψ' vanishes on the sub–local system of $\bigwedge^2 R^1p_*\mathbb{Z}$ corresponding to $(R^2q_*\mathbb{Z})_0$. Therefore, the image of ψ coincides with the image of $\mathbb{L}$ (after identifying $\bigwedge^2 R^1p_*\mathbb{Z}$ with $R^2q_*\mathbb{Z}$). Hence the rank of the image of ψ is at most one.

Let $\xi \in H_{6g-6}(\mathcal{M}_X, \mathbb{Z})$ be the homology class defined by the image of $\mathcal{N}_X$ by the map γ in (2.6). Let ω denote the first Chern class of the ample generator of $\mathrm{Pic}(\mathcal{N}_X) \cong \mathbb{Z}$. Since the homomorphism

$$\bigwedge^2 H^3(\mathcal{N}_X, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

defined by

$$\alpha \bigwedge \beta \longmapsto \int_{\mathcal{N}_X} \alpha \cup \beta \cup \omega^{3g-6}$$

is nonzero, we conclude that the composition

$$\bigwedge^2 H^3(\mathcal{M}_X, \mathbb{Z}) \xrightarrow{\psi|_{[X]}} H^{6g-6}(\mathcal{M}_X, \mathbb{Z}) \xrightarrow{\cap \xi} \mathbb{Z}$$

is nonzero. Hence the homomorphism ψ in (3.1) is nonzero. This completes the proof of the lemma. $\square$

For any $[X] \in M_g^0$, the homomorphism $\psi|_{[X]}$ gives an element

$$\theta \in \mathrm{Hom}(\bigwedge^2 H^3(\mathcal{M}_X, \mathbb{Z}), \mathbb{C}) = H^2(J^2(\mathcal{M}_X), \mathbb{C}) \tag{3.3}$$

up to a scalar multiplication. More precisely, θ is a complex line in $H^2(J^2(\mathcal{M}_X, \mathbb{C})$.

Theorem 3.5. *Let X and Y be smooth curves of genus $g \geq 3$. If $\mathcal{M}_X \cong \mathcal{M}_Y$, then $X \cong Y$.*

Proof. We have to show that it is possible to recover the pair $(J(X), \Theta)$ from $\mathcal{M}_X$. The result then follows from the classical Torelli theorem. By Corollary 3.2, the variety $\mathcal{M}_X$ determines $J(X)$ up to isomorphism.

In the proof of Lemma 3.4 we saw that ψ vanishes on $(R^2 q_* \mathbb{Z})_0$. Hence the complex line θ (defined in (3.3)) contains the canonical principal polarization.

We recall that a principal polarization ν on in abelian variety A is an element of $H^2(A, \mathbb{Z})$ which is ample and satisfies the condition $\nu^{\dim A} \cap [A] = 1$. In particular, the integral cohomology class ν is indivisible.

We have $\theta \cap H^2(J^2(\mathcal{M}_X), \mathbb{Z}) \neq 0$ (as θ contains the canonical principal polarization). Hence the $\mathbb{Z}$–module $\theta \cap H^2(J^2(\mathcal{M}_X), \mathbb{Z}) \cong \mathbb{Z}$ has two generators. Since the principal polarization is an ample class, exactly one of the two generators of $\theta \cap H^2(J^2(\mathcal{M}_X), \mathbb{Z})$ can be the principal polarization. Therefore, we have constructed the canonical principal polarization on $J^2(\mathcal{M}_X)$ from ψ. This completes the proof of the theorem. $\square$

Remark 3.6. It is not clear whether this argument can be generalized to bundles of arbitrary rank. The missing ingredient is an estimate on the dimension of $H^{\dim_{\mathbb{C}} \mathcal{M}_X}(\mathcal{M}_X, \mathbb{C})$.

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School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

E-mail address: indranil@math.tifr.res.in

Université Lille 1, Mathématiques – Bât. M2, F-59655 Villeneuve d'Ascq Cedex, France

E-mail address: nagel@math.univ-lille1.fr