A Lefschetz-type result for Koszul cohomology

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1 Introduction

Let X be a smooth, irreducible, projective variety defined over $\mathbb{C}$, let L be a line bundle on X and let $D \subset X$ be an effective, reduced and connected divisor. Set $M = \mathcal{O}_X(D)$. A natural problem related to this framework is to study the relationship between the Koszul cohomology groups $K_{p,q}(X, L)$ and $K_{p,q}(D, L_D)$.

In the case where D is smooth, M. Green [5, Theorem (3.b.1)] has proved, with the extra assumptions $H^0(X, L - M) = 0$ and $H^1(X, qL - M) = 0$ for all $q \geq 0$, that there exists a long exact sequence

$$\to K_{p,q}(X, -M, L) \to K_{p,q}(X, L) \to K_{p,q}(D, L_D) \to K_{p-1,q+1}(X, -M, L) \to$$

(connectedness actually follows directly from the vanishing conditions on the cohomology). Under the same smoothness hypothesis on D, another result of similar flavour was proved in [5, Theorem (3.b.7)]: if X is regular (i.e. $H^1(X, \mathcal{O}_X) = 0$), $L = M$, and $H^1(X, qL) = 0$ for all $q \geq 1$, then $K_{p,q}(X, L) \cong K_{p,q}(D, L_D)$, for all p and q.

A simple analysis of Green's proofs shows that smoothness of D is not necessary, and similar statements hold in the enlarged settings that we mentioned at the beginning, namely D to be reduced and connected.

In this paper we study the relationship between some of the Koszul groups of X and D under assumptions weaker than Green's ones. Our main result is the following.

Theorem 1.1. *Let X be a smooth, irreducible, projective, regular variety, let $D \neq 0$, and $E \neq 0$ be two effective divisors on X, and denote $d = h^0(D, \mathcal{O}_D(D))$, $e = h^0(E, \mathcal{O}_E(E))$, $\Delta = D + E$, and $L = \mathcal{O}_X(\Delta)$. Suppose that both D and Δ are reduced and connected. If $K_{p+1,1}(D, L_D) = 0$ for an integer $p > 0$ and one of the following two conditions is satisfied:*

(i) $e = 0$, *or*

(ii) $p \geq d$,

then we have an injective map $K_{p+e,1}(X, L) \to K_{p,1}(D, L_D)$.

In particular, we obtain the following Corollary.

Corollary 1.2. *Under the assumptions of Theorem 1.1, if moreover* $K_{p,1}(D, L_D) = 0$*, then* $K_{p+e,1}(X, L) = 0$.

We refer to Sections 4 and 5 for applications, and examples. They also witness for sharpness in the statement of Corollary 1.2.

One can enquire next about what happens when dropping the regularity condition. In the most general setting, we can prove the following slightly weaker result.

Proposition 1.3. *Let* X *be a smooth, irreducible, projective variety, and let* L *and* M *be line bundles on* X*. Let* $D \in |M|$ *be an effective, reduced, connected divisor, and set* $e = h^0(X, L - M)$*. Let* p *be a positive integer.*

(i) *If* $K_{p+1,1}(D, L_D) = 0$ *then we have an injective map* $K_{p+e,1}(X, L) \to K_{p,1}(D, L_D)$*;*

(ii) *If* $K_{p,1}(D, L_D) = 0$ *then* $K_{p+e,1}(X, L) = 0$.

Remark 1.4. The case $e = 0$ of Proposition 1.3 (ii) can be deduced from the long exact sequence obtained by Green [5, Thm. (3.b.1)]. But in our case one does not need the extra assumption $H^1(X, qL - M) = 0$, $q \geq 0$.

The outline of the paper is as follows. Section 2 is devoted to some general facts about base change and vanishing of Koszul cohomology for graded modules over a symmetric algebra. These results will be used in Section 3 for the proofs of Theorem 1.1 and of Proposition 1.3. In the last two sections, we present some applications.

Green's generic conjecture was used in [3] to prove Green-Lazarsfeld's gonality conjecture for generic curves of large gonality, with the sole exception of the generic gonality in the odd genus case. In Section 4, we try to go in the opposite direction, from the gonality conjecture to Green conjecture. We concentrate on curves on $K3$ surfaces, as they seem to be the most appropriate for proving generic syzygy conjectures. The attempt to have a better picture of the way the two conjectures were related was actually the original motivation for our work. For the moment, we are not able to give a full explanation of the phenomenon, and the connections between both conjectures seem to us to be much deeper than exposed here, or as found in [2] and [3].

The last application of Corollary 1.2 is a short discussion of the naturality of Voisin's method of proving Green's conjecture for generic curves of odd genus, see [15]; this is treated in Section 5.

2 Vanishing of Koszul cohomology and base change

Let V be a finite–dimensional $\mathbb{C}$–vector space and let B be a graded module over the symmetric algebra $S(V)$. We say that B satisfies hypothesis $(*)$ if

$$(*) \qquad B_q = 0 \text{ for all } q < 0 \text{ and } K_{p,0}(B,V) = 0 \text{ for all } p \geq 1.$$

Lemma 2.1.

(i) Let $W \subset V$ be a linear subspace. If B satisfies $(*)$, the natural map $K_{p,1}(B,W) \to K_{p,1}(B,V)$ is injective if $p > 0$.

(ii) If $B' \subset B$ is a graded $S(V)$–submodule and $B'_{q-1} = B_{q-1}$ then $K_{p,q}(B',V) \subset K_{p,q}(B,V)$ for all p.

Proof: For (i), choose a flag of linear subspaces

$$W = W_0 \subset W_1 \subset \ldots \subset W_c = V$$

such that $\dim(W_i/W_{i-1}) = 1$ for $i = 1, \ldots, c$. Set $L_i = W_i/W_{i-1}$. The short exact sequence

$$0 \to \bigwedge^p W_{i-1} \to \bigwedge^p W_i \to \bigwedge^{p-1} W_{i-1} \otimes L_i \to 0$$

induces an exact sequence

$$K_{p,0}(B,W_{i-1}) \otimes L_i \to K_{p,1}(B,W_{i-1}) \to K_{p,1}(B,W_i).$$

As $K_{p,0}(B,W_{i-1}) = 0$ for all $p > 0$ by hypothesis $(*)$, we obtain a chain of inclusions

$$K_{p,1}(B,W) = K_{p,1}(B,W_0) \subset \ldots \subset K_{p,1}(B,W_c) = K_{p,1}(B,V).$$

To prove (ii), set $B'' = B/B'$ and use the exact sequence

$$K_{p+1,q-1}(B'',V) \to K_{p,q}(B',V) \to K_{p,q}(B,V).$$

□

For every linear subspace $W \subset V$, and every $\ell \in \mathbb{Z}$, there exists a spectral sequence of change of base [6, Proposition (1.b.1)]

$$E_1^{p,q} = K_{-q,p+q}(B,W) \otimes \bigwedge^{\ell-p}(V/W) \Rightarrow K_{\ell-p-q,p+q}(B,V).$$

If B is a graded $S(V/W)$–module, B is an $S(V)$–module that is trivial as $S(W)$–module. In this case

$$E_2^{p,q} = \bigwedge^{-q} W \otimes K_{\ell-p,p+q}(B,V/W).$$

Lemma 2.2. *Let $W \subset V$ be a linear subspace of dimension e and let B be a graded $S(V/W)$–module. If B satisfies condition $(*)$ and $K_{p+1,1}(B,V/W) = 0$ for an integer $p > 0$, we have an isomorphism $K_{p,1}(B,V/W) \cong K_{p+e,1}(B,V)$.*

Proof: We use Green's spectral sequence mentioned above.

If $K_{p+1,1}(B,V/W) = 0$ then $K_{k,1}(B,V/W) = 0$ for all $k \geq p+1$, hence the terms $E_2^{i,1-i} = K_{\ell-i,1}(B,V/W) \otimes \bigwedge^{i-1} W$, where $\ell = p+e+1$, vanish if $i \neq e+1$. For $i = e+1$ we have $E_2^{e+1,-e} \cong E_\infty^{e+1,-e}$, as all the differentials arriving at and starting from $E_r^{e+1,-e}$ are zero for $r \geq 2$. Hence we obtain $E_\infty^1 = E_\infty^{e+1,-e}$, i.e., $K_{p+e,1}(B,V) \cong K_{p,1}(B,V/W) \otimes \bigwedge^e W \cong K_{p,1}(B,V/W)$. □

3 Proofs of main results

Proof of Theorem 1.1: We have exact sequences of sheaves (decomposition sequences, cf. [4, pp. 47–48])

$$0 \to \mathcal{O}_E(-D) \to \mathcal{O}_\Delta \to \mathcal{O}_D \to 0 \tag{1}$$

$$0 \to \mathcal{O}_D(-E) \to \mathcal{O}_\Delta \to \mathcal{O}_E \to 0. \tag{2}$$

Take the tensor product of the exact sequence (2) with $\mathcal{O}_\Delta(\Delta)$ to obtain an exact sequence

$$0 \to \mathcal{O}_E(E) \to L_\Delta \to L_D \to 0.$$

Set $W = H^0(E, \mathcal{O}_E(E))$, $V = H^0(\Delta, L_\Delta)$, $U = H^0(D, L_D)$ and

$$B^0 = \bigoplus_q H^0(E, L^{q-1} \otimes \mathcal{O}_E(E)),\ B^1 = \bigoplus_q H^0(\Delta, L_\Delta^q),$$

$$B = \bigoplus_q H^0(D, L_D^q),\ B^2 = B^1/B^0.$$

The exact sequence

$$0 \to W \to V \to U$$

induces an inclusion $V/W \subset U$. With the notation above, we have $K_{p,1}(B,U) = K_{p,1}(D,L_D)$. As

$$B_0 = \mathbb{C},\ B_1 = U,\ B_q = 0 \text{ for all } q < 0$$

the $S(U)$–module B satisfies hypothesis $(*)$. By Lemma 2.1 (i), we have an inclusion $K_{p,1}(B,V/W) \subset K_{p,1}(B,U)$. Using Lemma 2.2, we obtain an isomorphism $K_{p,1}(B,V/W) \cong K_{p+e,1}(B,V)$ (remark that vanishing of $K_{p+1,1}(B,U)$ implies vanishing of $K_{p+1,1}(B,V/W)$). Since $B^2_0 = B_0 = \mathbb{C}$ it follows from Lemma 2.1 (ii) that $K_{p+e,1}(B^2,V) \subset K_{p+e,1}(B,V)$, hence $K_{p+e,1}(B^2,V)$ injects into $K_{p,1}(B,U)$.

Regularity of X, and assumptions we have made on Δ, give rise to an isomorphism $K_{p+e,1}(X,L) \cong K_{p+e,1}(\Delta,L_\Delta)$. This result follows directly from the proof of [5, Theorem (3.b.7)] (see also [1, Remark 1.3]). By definition, $K_{p+e,1}(\Delta,L_\Delta) = K_{p+e,1}(B^1,V)$. The exact sequence of $S(V)$–modules

$$0 \to B^0 \to B^1 \to B^2 \to 0$$

induces an exact sequence

$$K_{p+e,1}(B^0,V) \to K_{p+e,1}(B^1,V) \to K_{p+e,1}(B^2,V).$$

To finish the proof, it suffices to show that $K_{p+e,1}(B^0,V) = 0$. The first case is clear: if $e = 0$ then $B^0_1 = 0$, hence $K_{p+e,1}(B^0,V) = 0$. To deal with the second case, we define $U' = H^0(E,L_E)$, $W' = H^0(D,\mathcal{O}_D(D))$ and consider the exact sequence

$$0 \to W' \to V \to V/W' \to 0.$$

If $K_{p+e-d,1}(B^0,V/W') = 0$, then $K_{p+e,1}(B^0,V) = 0$ by Lemma 2.2. Since $B^0_q = 0$ for all $q \le 0$, the $S(U')$–module B^0 satisfies hypothesis $(*)$. Hence the vanishing of $K_{p+e-d,1}(B^0,V/W')$ would follow from the vanishing of $K_{p+e-d,1}(B^0,U')$ using Lemma 2.1 (i). We identify

$$K_{p+e-d,1}(B^0,U') = K_{p+e-d,1}(E,\mathcal{O}_E(-D_E),L_E).$$

If $p \ge d$, then $p+e-d \ge e$, hence $K_{p+e-d,1}(E,\mathcal{O}_E(-D_E),L_E) = 0$ by Green's vanishing theorem [5, Theorem (3.a.1)]. □

Proof of Proposition 1.3: The proof of Theorem 1.1 goes through if we replace V by $H^0(X,L)$ and B^1 by $\oplus_q H^0(X,L^q)$ and note that $K_{p,1}(X,-M,L) = 0$ if $p \ge e$ by Green's vanishing theorem [5, Theorem (3.a.1)]. □

4 Relations between Green's conjecture and the gonality conjecture

Let L be a line bundle on a smooth projective variety X, and set $r = h^0(X, L) - 1$. Following [7, p. 87] we say that L satisfies property (M_k) if

$$K_{p,q}(X, L) = 0 \text{ for all } (p, q) \text{ such that } p \geq r - k \text{ and } q \neq 2.$$

Green and Lazarsfeld [7, Conjecture (3.7)] conjectured that for line bundles of sufficiently large degree on curves this property should be connected to the gonality of the curve.

Conjecture 4.1. (Green–Lazarsfeld's gonality conjecture, strong form) *Let C be a smooth curve of genus g, and let L be a line bundle on C. If $\deg L \geq 2g + k$ then L satisfies property (M_k) if and only if C does not carry a g^1_k.*

In the proofs of [1, Theorem 8.1], and [2, Theorem 2], one uses a vanishing of Koszul cohomology for some curves on a certain rational surface, compatible with the gonality conjecture, to prove Green's conjecture for other curves lying on the same surface. This suggests that there might be a very subtle connection between the two conjectures, at least for a generic choice curves in some gonality strata. We propose here a further investigation of this possibility.

Example 4.2 We consider a $K3$ surface S such that $\mathrm{Pic}(S) = \mathbb{Z}[D] \oplus \mathbb{Z}[F]$ with D a smooth, connected, curve of genus g on S such that $\mathrm{gon}(D) = k$ is computed by the restriction of an elliptic pencil $|F|$ on S. For all pairs (g, k) such that $g \geq 3$ and $2 \leq k \leq [\frac{g+3}{2}]$ such surfaces exist [9, Thm. 1.1 and Prop. 2.1].

Let F_1 and F_2 be two distinct elliptic curves in the elliptic pencil, and put $E = F_1 + F_2$, $\Delta = D + E$ and $L = \mathcal{O}_S(\Delta)$. We have $\deg(L_D) = 2g-2+2k$ and $h^0(D, L_D) = g - 1 + 2k$. As $\mathcal{O}_D(D) = K_D$, we have $d = h^0(D, \mathcal{O}_D(D)) = g$. Using [11, Prop. 2.6] we obtain $h^1(S, \mathcal{O}_S(2F)) = 1$ and $h^0(S, \mathcal{O}_S(2F)) = 3$, hence $e = 2$. As $\deg L_D \geq 2g + k$, the gonality conjecture predicts that $K_{p,1}(D, L_D) = 0$ if $p \geq h^0(D, L_D) - k = g - 1 + k$. Hence, if we assume that the gonality conjecture holds for D then $K_{p,1}(S, L) = 0$ for all $p \geq g + 1 + k$ by Corollary 1.2. As $D^2 > 0$ and D is irreducible, the linear system $|D|$ is base–point free [11, Thm. 3.1]. The linear system $|D + 2F|$ obtained by adding a multiple of the base–point free pencil $|F|$ is then also base–point free and contains a smooth, irreducible curve Γ by Bertini's theorem. Its genus is $\gamma = g + 2k$, and its gonality and Clifford index are $\mathrm{gon}(\Gamma) = k$, $\mathrm{Cliff}(\Gamma) = k - 2$ (cf. [9, Proof of Prop. 2.1]). Using [5, Theorem (3.b.7)] we obtain $K_{p,1}(\Gamma, K_\Gamma) = 0$ for all $p \geq \gamma - k + 1$. By Green's duality theorem [5, Corollary (2.C.10)] we have $K_{p,1}(\Gamma, K_\Gamma)^\vee \cong K_{\gamma-p-2,2}(\Gamma, K_\Gamma)$. Hence

$$K_{p,2}(\Gamma, K_\Gamma) = 0 \text{ if } p \leq k - 3.$$

This is exactly the vanishing predicted by Green's conjecture for the curve Γ. As it is known that $K_{k-2,2}(\Gamma, K_\Gamma) \neq 0$ [5, Appendix], this example shows that Corollary 1.2 is sharp for $e > 0$.

Remark 4.3 The conclusion "gonality conjecture for $D \Rightarrow$ Green's conjecture for Γ" of Example 4.2 remains true if D is a smooth, connected curve on a K3 surface S such that the gonality of D is computed by an elliptic pencil $|F|$ on S. For example, such an elliptic pencil on S also exists if the Clifford index of D is small compared to the genus; see [10, Theorem 3.5].

5 A comment on Voisin's proof of Green's conjecture

In two recent papers [14], [15], C. Voisin realised a major breakthrough in the theory of syzygies of curves by proving Green's conjecture for generic curves. Using a completely new way of computing Koszul cohomology spaces, she verified Green's conjecture for curves lying on some $K3$ surfaces. We try to explain, using our Corollary 1.2, why her choice in [15] is a natural one.

In [15] Voisin considers a $K3$ surface S whose Picard group is generated by a very ample divisor D, where D is a smooth curve of odd genus $2k+1$, and by a smooth rational curve E which intersects D in two distinct points p and q. If Green's conjecture is true for generic curves of odd genus, then it holds for any curve of odd genus and maximal gonality [8]. In particular, Green's conjecture then holds for D and $K_{k,1}(D, K_D) = 0$. Using Theorem 3 of [1], this would give $K_{k+1,1}(D, K_D + p + q) = 0$. Since $K_D + p + q$ is the restriction of the line bundle $L = \mathcal{O}_S(D + E)$ we can apply Corollary 1.2, with $e = 0$, to obtain $K_{k+1,1}(S, \mathcal{O}_S(D+E)) = 0$. So we obtain the following statement.

(∗∗) If Green's conjecture holds for the generic curve of odd genus, then $K_{k+1,1}(S, \mathcal{O}_S(D + E)) = 0$.

The first step in Voisin's proof is to prove $K_{k+1,1}(S, \mathcal{O}_S(D + E)) = 0$, using methods similar to those in [14]; see [15, Thm.5]. In the second step she proves the converse of the above statement $(**)$ to obtain Green's conjecture for generic curves of odd genus.

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