Cohomology of quadric bundles

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Contents

Résumé Français

Cette thèse d'habilitation contient une description d'une partie de mes activités de recherche depuis ma thèse de doctorat.

Après ma thèse j'ai étudié notamment le théorème de connectivité de Nori [56]. Ce résultat est un théorème du type de Lefschetz pour la cohomologie de la famille universelle des intersections complètes dans une variété projective. L'énoncé original de Nori est un résultat asymptotique, valable pour des intersections complètes de multidegré suffisamment élevé.

Dans [51] j'ai obtenu une version effective de ce théorème. Les exposés [52] contiennent une description détaillée de mes travaux sur le théorème de Nori. Pour cette raison j'ai décidé d'écrire un mémoire sur la cohomologie des fibrés en quadriques au lieu de faire une synthèse de tous mes travaux depuis la thèse.

L'étude de l'optimalité des bornes obtenues dans [51] m'a amené naturellement à regarder des fibrés en quadriques. En effet, la plupart des exemples en bas degré sont des intersections complètes de quadriques. Pour une description explicite de leur cohomologie on passe au fibré en quadriques associé à cette intersection complète.

Cependant, il y a une deuxième relation entre le théorème de Nori et les fibrés en quadriques. L'une des questions évoquées dans [56] est de savoir s'il est possible d'obtenir le théorème de connectivité par des méthodes topologiques (monodromie, étude de la cohomologie à valeurs dans un système local) et des arguments de poids issus de la théorie de Hodge mixte. Une telle approche permettrait par exemple d'obtenir un analogue du théorème de connectivité de Nori pour la cohomologie ℓ–adique. Cette approche est discutée dans l'introduction; elle mène à une étude de la dégènerescence de la suite spectrale de Leray. En général on a peu d'espoir de pouvoir mener à bien cette étude, mais si on réduit la taille de la base (au lieu de consdiérer la famille universelle on regarde un sous–espace de basse dimension) on peut obtenir des résultats partiels.

Le cas des pinceaux de Lefschetz a été étudié par N. Katz [41]. Pour des fibrés en quadriques il est possible d'étudier la suite spectrale de Leray pour des bases de dimension supérieure à l'aide de la stratification naturelle de la base donnée définie par le corang des quadriques et la théorie de Hodge mixte. Après avoir rappelé un nombre de résultats basiques sur les fibrés en quadriques et des théorèmes d'annulation pour la cohomologie à valeurs dans un sysème local, nous étudions la suite spectrale de Leray associée à un fibré en quadriques dans le Chapitre 2. Ceci nous permet de redémontrer d'une façon systématique un certain nombre de résultats classiques sur les

fibrés en quadriques obtenus dans [60], [8], [43], [58], [48]. Dans le dernier chapitre on étudie la cohomologie des fibrés en quadriques munis d'une involution. La motivation pour ce chapitre était un exemple de Bardelli [6]. Dans ce chapitre nous discutons aussi le lien avec le théorème de Nori et des applications géométriques concernant l'image de l'application d'Abel–Jacobi et l'application régulateur.

Publications

[1] The Abel–Jacobi map for complete intersections, Indag. Math. **8** (1997), 95–113. (publication issue de la thèse)

[2] The generalized Hodge conjecture for the quadratic complex of lines in projective four–space, Math. Ann. **312** (1998), 387–401. (publication issue de la thèse)

[3] Effective bounds for Nori's connectivity theorem, Comptes Rendus Acad. Sci. Paris, Série I (1998), 189–192.

[4] A variant of a theorem of C. Voisin, Indag. Math. **12** (2001), 231–241.

[5] Effective bounds for Hodge–theoretic connectivity, J. Alg. Geom. **11** (2002), 1–32.

[6] Lectures on Nori's connectivity theorem. Transcendental aspects of algebraic cycles (C. Peters, S. Müller–Stach eds.) London Math. Soc. Lecture Note Ser., 313, Cambridge Univ. Press, Cambridge (2004), 235–275.

[7] (avec M. Aprodu) A Lefschetz type result for Koszul cohomology. Manuscripta Math. 114 (2004), 423–430.

[8] The regulator map for complete intersections, à paraître dans Algebraic Cycles and Motives, Proceedings Eager conference 2004.

[9] (avec I. Biswas) A Torelli type theorem for the moduli space of rank two connections on a curve, à paraître dans Comptes Rendus Acad. Sci. Paris.

Chapter 1

Introduction

This habilitation thesis contains a description of a part of the work done after my PhD thesis. The main result that I obtained during this period is an effective version of Nori's connectivity theorem [51]. The lecture notes [52] contain a detailed exposition of this result and its geometric applications. In this thesis I decided to concentrate on more recent work, which grew out of an attempt to understand whether the degree bounds obtained in [51] are optimal for complete intersections in projective space. This naturally leads to the study of Abel–Jacobi and regulator maps on complete intersections of quadrics, and to the study of the cohomology of quadric bundles. This connection is explained in more detail in Chapter 4.

In the remainder of the introduction I would like to discuss another connection between Nori's theorem and quadric bundles, which motivates the techniques that we shall use and puts them in a more general perspective.

Let us first recall the statement of Nori's connectivity theorem [56].

Theorem 1.0.1 (Nori) *Let $(Y, \mathcal{O}_Y(1))$ be a smooth polarised variety. Given positive integers $d_0, \ldots, d_s$, we consider the vector bundle*

$$E = \mathcal{O}_Y(d_0) \oplus \ldots \oplus \mathcal{O}_Y(d_s)$$

and put $S = \mathbb{P}H^0(Y, E)$. Let $X_S \to S$ be the universal family of complete intersections in Y of type $(d_0, \ldots, d_s)$. Let $\Delta \subset S$ be the discriminant locus, and put $U = S \setminus \Delta$. Let n be the relative dimension of $X_U = X \times_S U$ over U, and let $T \to U$ be a smooth morphism. If $\min(d_0, \ldots, d_s) \gg 0$ then $H^{n+k}(Y \times T, X \times_S T, \mathbb{Q}) = 0$ for all $k \leq n$.

Since the morphism $f_T : X_T \to T$ is smooth, the Leray spectral sequence

$$E_2^{p,q}(f_T) = H^p(T, R^q f_{T,*}\mathbb{Q}) \Rightarrow H^{p+q}(X_T, \mathbb{Q})$$

degenerates at E_2 by a theorem of Deligne [26]. Let $p_2 : Y \times T \to T$ be the projection onto the second factor. The inclusion map $i : X_T \to Y \times T$ induces a homomorphism of local systems

$$i^* : R^n p_{2,*}\mathbb{Q} \to R^n f_{T,*}\mathbb{Q}$$

which is injective by the Lefschetz hyperplane theorem. Put $\mathbb{V} = \operatorname{coker} i^*$, and let us assume for simplicity that Y is a projective space. In this case Nori's theorem is equivalent to the following statement; see [56, §4].

$$H^k(T, \mathbb{V}) = 0 \quad \text{for all } k < n. \tag{1.1}$$

In Nori's paper and subsequent work on the subject this statement is proved using infinitesimal techniques and mixed Hodge theory.

As Nori points out [56, 7.3], the analogue of Theorem 1.0.1 for ℓ–adic cohomology is not known if the varieties in question are defined over a field of positive characteristic. Hence it would be desirable to have a geometric proof that uses only monodromy techniques and weight considerations. A first, somewhat naive apporach could be the following. The full universal family $X_S \subset \mathbb{P}^N \times S$ is the zero locus of a section of an ample vector bundle on $\mathbb{P}^N \times S$; cf. [52, Remark 7.4.2 (ii)]. Hence

$$H^q(\mathbb{P}^N \times S, X_S, \mathbb{Z}) = 0 \;\; \text{for all k} \leq \mathrm{dim\ X_S} \tag{1.2}$$

by the Lefschetz theorem [44]. Hence the cohomology of X_S is known in a range that exceeds the range of Nori's connectivity theorem, and we could try to obtain information on $H^*(X_U)$ by comparing it to $H^*(X_S)$. Let $j : U \to S$ be the inclusion map, and consider the adjunction homomorphism

$$\nu : R^n f_{S,*}\mathbb{Q} \to j_*j^*R^n f_{S,*}\mathbb{Q} = R^n f_{U,*}\mathbb{Q}.$$

Suppose that ν induces an isomorphism

$$E_2^{k,n}(f_S) \cong E_2^{k,n}(f_U) \tag{1.3}$$

for all $k \geq 0$. If furthermore

$$E_\infty^{k,n}(f_S) = E_2^{k,n}(f_S) \tag{1.4}$$

then $E_2^{k,n}(f_U) \cong E_\infty^{k,n}(f_S)$ comes from the cohomology of $\mathbb{P}^N \times S$, and the vanishing (1.1) follows. Hence, if condition (1.3) is satisfied the vanishing (1.1) can be reduced to an analysis of the Leray spectral sequence for f_S. This requires a more detailed analysis of the constructible sheaves $R^q f_*\mathbb{Q}$. There exist a stratification

$$S = \Delta_0 \supset \Delta_1 \supset \Delta_2 \supset \ldots \supset \Delta_M \supset \Delta_{M+1} = \emptyset$$

and a finite filtration $F^\bullet$ on $R^qf_*\mathbb{Q}$ such that

$$\mathrm{Gr}_F^\alpha R^qf_*\mathbb{Q} \cong j_{\alpha,!}\mathbb{L}_\alpha$$

with $j_\alpha : U_\alpha \to S$ the inclusion of the locally closed subset $U_\alpha = \Delta_\alpha \setminus \Delta_{\alpha+1}$ and $\mathbb{L}_\alpha$ a local system on U_α. In principle one should then be able to compute the terms $E_2^{p,q}(f_S)$ from the cohomology groups

$$H^p(U_\alpha, j_{\alpha,!}, \mathbb{L}_\alpha) = H_c^p(U_\alpha, \mathbb{L}_\alpha).$$

This knowledge could then be used to deduce the vanishing of differentials in the Leray spectral sequence using vanishing theorems for cohomology with values in a local system and/or weight arguments.

The above discussion only serves as a guideline; it will be very difficult to carry out the above procedure in practice. To simplify the original problem one can reduce the dimension of the base space. Consider for example the following setup. Take $Y = \mathbb{P}^{n+1}$, $S = \mathbb{P}H^0(\mathbb{P}^{n+1}, \mathcal{O}_\mathbb{P}(d))$. Choose a linear subspace $\mathbb{P}^r \subset S$ that is transversal to the stratification $\{\Delta_\alpha\}$ and consider the pullback

$$\mathcal{X} = X_S \times_S \mathbb{P}^r \to \mathbb{P}^r$$

with the induced stratification. In this case the analogue of (1.2) fails; the cokernel of the restriction map

$$i^* : H^{n+r}(\mathbb{P}^{n+1} \times \mathbb{P}^r) \to H^{n+r}(\mathcal{X})$$

is isomorphic to $H_{\mathrm{pr}}^{n-r}(X)$, the primitive cohomology of the base locus X of the linear system. This result is known as the *Cayley trick*; see Proposition 2.2.10.

The case $r = 1$ was studied in detail by N. Katz [41]. He introduced the following condition [loc. cit., 5.3].

$$\nu : R^qf_{S,*}\mathbb{Q} \to \ j_*R^qf_{U,*}\mathbb{Q} \ \text{ is an isomorphism for all } q \geq 0. \tag{1.5}$$

This condition clearly implies (1.3), and for Lefschetz pencils it also implies (1.4) [loc. cit., Thm. 5.6].

For $r \geq 2$ little seems to be known in general. The only case where the stratification $\{\Delta_\alpha\}$ and the local systems L_α are well understood is the case $d = 2$. To analyse the Leray spectral sequence in this case, we shall first quotient out the 'fixed' part of the sheaves $\mathbb{L}_\alpha$ and the corresponding E_2 terms to obtain a spectral sequence (the subscript 'v' and the notation $\mathbb{V}_\alpha$ stand for 'variable')

$$E_2^{p,q}(f)_v = H_c^p(U_\alpha, \mathbb{V}_\alpha) \Rightarrow H^{p+q}(\mathcal{X}, \mathbb{Z})_v. \tag{1.6}$$

The existence of this spectral sequence requires some results on mixed Hodge theory; see Theorem 2.3.5. For low values of r, we can completely analyse the behaviour of this spectral sequence. It turns out that there is one single E_2 term that survives to E_∞. This method gives a unified proof of a number of classical results on the cohomology of quadric bundles [60], [8], [43], [58], [48] that had previously been obtained using a variety of different techniques.

As r increases it becomes harder to analyse the precise behaviour of the spectral sequence (1.6), since more and more nonzero terms appear and the mixed Hodge structure on $E_2^{p,q}(f)_v$ Still it is possible to show that for a generic quadric bundle only one E_2 terms survives to E_∞ using a degeneration argument due to Terasoma [66]. The situation somewhat improves if we impose some extra structure on the quadric bundle. Motivated by a beautiful example of Bardelli [6] we study quadric bundles that are equipped with a nontrivial involution in Chapter 4.

This thesis is organised as follows. In Chapter 2 we recall a number of basic results on quadric bundles, cohomology of local systems and spectral sequences that will be needed in the sequel. In Chapter 3 we analyse the Leray spectral sequence associated to a quadric bundle and reprove a number of classical results from this point of view, in a slightly more general setting. Chapter 4 is devoted to quadric bundles equipped with an involution. We generalise a result of Bardelli and present some applications concerning the image of the Abel–Jacobi and regulator maps for complete intersections of quadrics.

Notation and conventions. We shall work throughout with varieties that are defined over the field of complex numbers. As we use only geometric arguments and weight considerations from mixed Hodge theory, most results should remain valid for ℓ–adic cohomology of varieties defined over a field of characteristic $p \neq 2$.

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Chapter 2

Preliminaries

2.1 Quadric bundles

Definition 2.1.1 *Let $\mathcal{X}$ and S be smooth, connected complex projective varieties. We say that $\mathcal{X}$ is a quadric bundle of relative dimension n over S if there exists a morphism $f : \mathcal{X} \to S$ such that the fibers of f are n–dimensional quadrics.*

Remark 2.1.2 Note that the hypotheses on f imply that f is proper and flat. If $n = 1$, we say that $\mathcal{X}$ is a *conic bundle*.

Definition 2.1.3 *Let X be a smooth, projective variety over $\mathbb{C}$, and let E be a vector bundle over X. A quadratic form on E with values in a line bundle L is a global section $q \in H^0(X, S^2E^\vee \otimes L)$.*

Proposition 2.1.4 (Beauville) *Let $f : \mathcal{X} \to S$ be a quadric bundle of relative dimension n. There exist a rank $n+2$ vector bundle E, a line bundle L, and a quadratic form $q \in H^0(S, S^2E^\vee \otimes L)$ such that $\mathcal{X}$ is the zero scheme of q in the projective bundle $\mathbb{P}(E)$.*

Proof: The proof proceeds in two steps.

Step 1. One first shows the existence of a line bundle A on $\mathcal{X}$ such that $A|_{\mathcal{X}_s} \cong \mathcal{O}_{\mathcal{X}_s}(1)$ for all $s \in S$. If $n = 1$ one takes $A = \omega_{\mathcal{X}/S}$, the relative dualising sheaf; if $n = 0$ one simply takes $A = \mathcal{O}_{\mathcal{X}}$.

For $n \geq 2$ one argues as follows. As S is smooth and connected, it is irreducible. Let η be the generic point of S, and let $\mathcal{X}_\eta \to \operatorname{Spec} k(\eta)$ be the generic fiber. As $\mathcal{X}$ is smooth, the inverse image $\mathcal{X}_U = f^{-1}(U)$ is smooth for every open Zariski open subset $U \subset S$ and the restriction map

$\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_U)$ is surjective by [40, II, Prop. 6.5 and Cor. 6.16]. Hence the natural map

$$\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(X_\eta) = \varinjlim_{U \subset S} \mathrm{Pic}(\mathcal{X}_U)$$

is surjective. This implies that the line bundle $\mathcal{O}_{\mathcal{X}_\eta}(1)$ lifts to a line bundle $A \in \mathrm{Pic}(\mathcal{X})$. As in [8, Prop. 1.2] one shows that A satisfies the required property.

Step 2. To finish the proof, one argues as in [27, 2.5]. Put $E = (f_* A)^\vee$. Flatness of f implies that E is a locally free sheaf of rank $h^0(\mathcal{X}_s, \mathcal{O}_{\mathcal{X}_s}(1)) = n+2$. Since the restriction of A to every fiber is globally generated, the adjunction homomorphism

$$f^* f_* A \to A$$

is surjective. The resulting closed immersion $\mathbb{P}(A^\vee) \hookrightarrow \mathbb{P}(f^* E)$ descends to a closed immersion $\mathcal{X} \to \mathbb{P}(E)$ that realises $\mathcal{X}$ as a relative divisor of degree 2 over S. Let ξ_E be the tautological line bundle on $\mathbb{P}(E)$, and let $p : \mathbb{P}(E) \to S$ be the projection map. As

$$\mathrm{Pic}(\mathbb{P}(E)) \cong p^* \mathrm{Pic}(S) \oplus \mathbb{Z}[\xi_E]$$

there exists a line bundle $L \in \mathrm{Pic}(S)$ such that $\mathcal{X} \in |\xi_E^2 \otimes p^* L|$, and the proof is finished using the isomorphism

$$H^0(\mathbb{P}(E), \xi_E^2 \otimes p^* L) \cong H^0(S, S^2 E^\vee \otimes L).$$

$\square$

Example 2.1.5 We list several examples of quadric bundles.

(i) Consider a linear system of quadrics $\langle Q_0, \ldots, Q_r \rangle \subset \mathbb{P}H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}}(2))$, and put

$$\mathcal{X} = \{(x, \lambda) \in \mathbb{P}^{n+1} \times \mathbb{P}^r \mid \lambda_0 Q_0(x) + \ldots + \lambda_r Q_r(x) = 0\}.$$

Projection to the second factor gives $\mathcal{X}$ the structure of a quadric bundle of relative dimension n.

(ii) Let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic hypersurface. Suppose that X contains a linear subspace $\Lambda \cong \mathbb{P}^s$, $s \geq 1$. Resolving the indeterminacy of the projection $\pi : X - - > \mathbb{P}^{n-s}$ with center Λ, we obtain a morphism

$$f : \tilde{X} = \mathrm{Bl}_\Lambda(X) \to \mathbb{P}^{n-s}$$

that gives $\tilde{X}$ the structure of a quadric bundle of relative dimension s; cf. [19] and [69].

(iii) Let $X = V(2,3) \subset \mathbb{P}^{n+1}$ be a smooth complete intersection of a quadric Q and a cubic F. If X contains a linear subspace $\Lambda \cong \mathbb{P}^s$, the projection from Λ induces a morphism

$$f : \tilde{X} = \mathrm{Bl}_\Lambda(X) \to \mathbb{P}^{n-s+1}$$

that makes $\tilde{X}$ into a quadric bundle of relative dimension $s-1$. This can be seen as follows. Given $x \in \mathbb{P}^{n-s+1}$, let $\Lambda_x = \langle \Lambda, x\rangle$ be the linear subspace of dimension $s+1$ spanned by Λ and x and write

$$\Lambda_x \cap Q = \Lambda \cup \Lambda'_x, \quad \Lambda_x \cap F = \Lambda \cup Q'_x.$$

The residual intersection $Q_x = \Lambda'_x \cap Q'_x$ is a quadric of dimension $s-1$.

(iv) Let $Q \subset \mathbb{P}^{n+1}$ be a smooth quadric, and let $G = G_2(\mathbb{P}^{n+1})$ be the Grassmann variety of planes in $\mathbb{P}^{n+1}$. Consider the incidence correspondence

$$\mathcal{X} = \{(x, \Lambda) \in Q \times G \mid x \in \Lambda\}.$$

Projection to the second factor gives $\mathcal{X}$ the structure of a conic bundle over G.

2.2 Cohomology of quadrics

Recall that the *rank* of a quadric $Q = V(\sum_{i,j} a_{ij}X_iX_j) \subset \mathbb{P}^{n+1}$ is the rank of the symmetric matrix (a_{ij}); its *corank* is defined by

$$\mathrm{corank}\,(Q) = n+2-\mathrm{rank}\,(Q).$$

Proposition 2.2.1 *Let $Q \subset \mathbb{P}^{n+1}$ be a quadric of corank s. Then*

$$H^k(Q,\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if} \quad k \neq n+s \text{ even} \\ \mathbb{Z}^2 & \text{if} \quad k = n+s \text{ even} \\ 0 & \text{if} \quad\quad k \text{ odd.} \end{cases}$$

Proof: A quadric of corank s is a projective cone with vertex $\Sigma = \mathbb{P}^{s-1}$ over a smooth quadric $Q' \subset \mathbb{P}^{n-s+1}$. Put $U = Q \setminus \Sigma$, and consider the exact sequence

$$H^k_c(U,\mathbb{Z}) \to H^k(Q,\mathbb{Z}) \to H^k(\Sigma,\mathbb{Z}) \to H^{k+1}_c(U,\mathbb{Z}).$$

Projection from Σ induces a morphism $p : U \to Q'$ which gives U the structure of a vector bundle over rank s over Q'. (More precisely, U is the total space of the vector bundle $E = \oplus^s \mathcal{O}_{Q'}(1)$.) Hence $p^* : H^{k-2s}(Q',\mathbb{Z}) \to$

$H^k_c(U,\mathbb{Z})$ is an isomorphism for all $k \geq 2s$, and $H^k_c(U,\mathbb{Z}) = 0$ if $k < 2s$. As $H^k(\Sigma,\mathbb{Z}) \neq 0 \iff k \leq 2s-2$, the previous exact sequence shows that

$$H^k(Q,\mathbb{Z}) = \begin{cases} H^k(\Sigma,\mathbb{Z}) & \text{if } 0 \leq k \leq 2s-2 \\ 0 & k = 2s-1 \\ H^{k-2s}(Q',\mathbb{Z}) & \text{if 2s} \leq \text{k} \leq \text{2n.} \end{cases}$$

As the cohomology of the smooth quadric Q' is known via its algebraic cell decomposition, the statement follows. □

Remark 2.2.2 The previous result could also be proved by constructing an algebraic cell decomposition for singular quadrics, starting from the cell decomposition of a smooth quadric. The proof given above has the advantage that it generalises to the relative situation; see Proposition 2.3.3.

Let $f : X \to Y$ be a proper morphism of algebraic varieties. Put $d_X = \dim_{\mathbb{C}}(X)$, $d_Y = \dim_{\mathbb{C}}(Y)$. If X is proper and Y is smooth, there exists a Gysin homomorphism

$$f_* : H^k(X,\mathbb{Z}) \to H^{k+2(d_Y-d_X)}(Y,\mathbb{Z})$$

which is a morphism of mixed Hodge structures of type (d_Y-d_X, d_Y-d_X); cf. [25]. If $(Y, \mathcal{O}_Y(1))$ is a smooth polarised variety and $X = V(d_0,\ldots,d_r) \subset Y$ is an n–dimensional complete intersection of multidegree $(d_0,\ldots,d_r)$ with inclusion map $i : X \to Y$, we obtain a Gysin map $i_* : H^k(X,\mathbb{Z}) \to H^{k+2r+2}(Y,\mathbb{Z})$.

Following [27, p. 17] we define two versions of primitive cohomology with $\mathbb{Z}$–coefficients.

Definition 2.2.3 *Notation as above.*

(i) The primitive part *of $H^k(X,\mathbb{Z})$ is the subgroup*

$$H^k(X,\mathbb{Z})_0 = \ker i_* : H^k(X,\mathbb{Z}) \to H^{k+2r+2}(Y,\mathbb{Z}).$$

(ii) The primitive quotient *of $H^k(X,\mathbb{Z})$ is the group*

$$H^k(X,\mathbb{Z})_v = \text{coker } i^* : H^k(Y,\mathbb{Z}) \to H^k(X,\mathbb{Z}).$$

Remark 2.2.4 The subscript 'v' in the part (ii) of the definition stands for 'variable'. Using the Lefschetz theorems one can show that the primitive part is related to the primitive cohomology in the sense of Lefschetz by the formula

$$H^k(X,\mathbb{Z})_0 \otimes \mathbb{Q} \subseteq H^k(X,\mathbb{Q})_{\text{pr}}.$$

Equality holds if Y is a projective space or, more generally, if $H^*(Y,\mathbb{Q})_{\text{pr}} = 0$.

Lemma 2.2.5 *Supppose that X is smooth. Then $H^n(X,\mathbb{Z})_0$ is the orthogonal complement of* $\operatorname{im} i^*$ *with respect to the intersection form on X. If* $\operatorname{im} i^*$ *is a primitive sublattice of $H^n(X,\mathbb{Z})$, then $H^n(X,\mathbb{Z})_0 \cong H^n(X,\mathbb{Z})_v^\vee$.*

Proof: The first assertion follows from the formula

$$(i^*a, b)_X = (a, i_*b)_Y$$

and the nondegeneracy of the pairing $(\,,)_X$. The second part is a standard result about lattices: if L is a unimodular lattice and $M \subset L$ is a primitive sublattice, the pairing on L induces a unimodular pairing $M^\perp \times L/M \to \mathbb{Z}$, hence an isomorphism $M^\perp \cong (L/M)^\vee$. $\square$

Proposition 2.2.6 *Let $Q \subset \mathbb{P}^{n+1}$ be a quadric of corank s. We have*

$$\begin{aligned} H^{2k}(Q,\mathbb{Z})_0 &\neq 0 \iff 2k = n+s \\ H^{2k}(Q,\mathbb{Z})_v &= \begin{cases} 0 & 2k < n+s \\ \mathbb{Z}/2 & 2k > n+s \end{cases} \end{aligned}$$

If $n+s$ is even, $H^{n+s}(Q,\mathbb{Z})_0$ and $H^{n+s}(Q,\mathbb{Z})_v$ are free abelian groups of rank one.

Proof: We first treat the case where Q is smooth. The Lefschetz hyperplane theorem implies that $H^{2k}(Q,\mathbb{Z})_0 = H^{2k}(Q,\mathbb{Z})_v = 0$ if $2k < n$. If $2k > n$, Poincaré duality identifies $H^{2k}(Q,\mathbb{Z})_0$ with the kernel of the map $i_* : H_{2n-2k}(Q,\mathbb{Z}) \to H_{2n-2k}(\mathbb{P}^{n+1},\mathbb{Z})$. This map is an isomorphism, again by the Lefschetz hyperplane theorem.

Let $[H] \in H^2(\mathbb{P}^{n+1},\mathbb{Z})$ be the class of a hyperplane, and let α be a generator of $H^{2k}(Q,\mathbb{Z})$. If $2k > n$ then $i^*[H]^{n-k}.\alpha = 2$, hence the image of the restriction map $i^* : H^{2k}(\mathbb{P}^{n+1},\mathbb{Z}) \to H^{2k}(Q,\mathbb{Z})$ is a subgroup of index 2.

The general case $\operatorname{corank}(Q) = s$ is reduced to the smooth case using Proposition 2.2.1. $\square$

Recall that the *level* of a Hodge structure H is defined by

$$\ell(H) = \max\{|p-q| \mid H^{p,q} \neq 0\}.$$

Proposition 2.2.7 *Let $\mathcal{X} \to S$ be a quadric bundle of relative dimension n defined by a quadratic form $q \in H^0(S, S^2E^\vee \otimes L)$. Let ξ_E be the tautological line bundle on $P = \mathbb{P}(E)$, and put $M = \xi_E^2 \otimes \pi^*L$. Suppose that the projective bundle P satisfies the following condition.*

$$H^i(P, \Omega_P^j \otimes M^k) = 0 \ \forall i > 0, \forall j \geq 0, \forall k > 0. \tag{2.1}$$

Put $r = \dim(S)$. Then

$$\begin{array}{rcll} \ell(H^{n+r}(\mathcal{X})_v \otimes \mathbb{C}) & \leq & r & \text{if } n \text{ is even} \\ \ell(H^{n+r}(\mathcal{X})_v \otimes \mathbb{C}) & \leq & r-1 & \text{if } n \text{ is odd.} \end{array}$$

Proof: The theory of Jacobi modules shows that $H^{n+r-q,q}(\mathcal{X})_v$ is a quotient of $H^0(P, K_P \otimes M^{q+1})$; cf. [38], [16, 10.4]. Let $\pi : P \to S$ be the projection map. Using the isomorphism $K_P \cong \pi^*(K_S \otimes \det E) \otimes \xi_E^{-n-2}$ and the projection formula, we obtain

$$H^0(P, K_P \otimes M^{q+1}) \cong H^0(S, K_S \otimes \det E \otimes L^{q+1} \otimes \pi_* \xi_E^{2q-n}).$$

Hence $H^{n+r-q,q}(\mathcal{X})_v = 0$ if $2q < n$. Write $n = 2m - \varepsilon$, $\varepsilon \in \{0, 1\}$. Then

$$2q < n \iff q < m \iff n + r - 2q > r - \varepsilon.$$

Hence $\ell(H^{n+r}(\mathcal{X}, \mathbb{C})_v) \leq r - \varepsilon$. □

Remark 2.2.8 The Bott–Danilov–Steenbrink vanishing theorem [7, Thm. 7.1] implies that condition (2.1) is satisfied if P is a toric variety and M is ample.

Remark 2.2.9 The generalised Hodge conjecture predicts that $H^{n+r}(\mathcal{X}, \mathbb{Q})$ should be supported on a subvariety $Z \subset \mathcal{X}$ of codimension $p \geq m$ for quadric bundles that satisfy the condition of Proposition 2.2.7. We shall verify the generalised Hodge conjecture for certain types of quadric bundles in Chapter 3.

The following result is known for $\mathbb{Q}$–coefficients and appears in several places; see [24, §6] for a nice discussion. We shall need a slightly more precise version, valid for $\mathbb{Z}$–coefficients.

Proposition 2.2.10 (Cayley trick) *Let $(Y, \mathcal{O}_Y(1))$ be a smooth polarised variety, and let $X = V(f_0, \ldots, f_r)$ be an n–dimensional complete intersection of multidegree $(d_0, \ldots, d_r)$ in Y. Put $E = \mathcal{O}_Y(d_0) \oplus \ldots \oplus \mathcal{O}_Y(d_r)$, $P = \mathbb{P}(E^\vee)$. The section $s = (f_0, \ldots, f_r) \in H^0(Y, E)$ corresponds to a section $\sigma \in H^0(P, \xi_E)$. Write $\mathcal{X} = V(\sigma) \subset P$. Suppose that*

(i) $H^*(Y, \mathbb{Z})$ is torsion free;

(ii) The Lefschetz operator $H^{n-1}(Y, \mathbb{Z}) \xrightarrow{L_Y} H^{n+1}(Y, \mathbb{Z})$ defined by $L_Y(\alpha) = \alpha \cup c_1(\mathcal{O}_Y(1))$ is an isomorphism.

Then we have an isomorphism of Hodge structures

$$H^n(X,\mathbb{Z})_v \cong H^{n+2r}(\mathcal{X},\mathbb{Z})_v(r).$$

Proof: The projection map $\varphi : P \to Y$ induces a morphism $P \setminus \mathcal{X} \to Y \setminus X$, which is an affine fiber bundle with fiber $\mathbb{A}^r$. Hence we obtain an isomorphism of Hodge structures $H_c^{n+1}(Y \setminus X, \mathbb{Z}) \cong H_c^{n+2r+1}(P \setminus \mathcal{X}, \mathbb{Z})(r)$. Put $U = Y \setminus X$, and let $i : X \to Y$, $j : U \to Y$ be the inclusion maps. As $H_c^{n+1}(U) = H^{n+1}(Y, j_!\mathbb{Z})$, the exact sequence

$$0 \to j_!\mathbb{Z} \to \mathbb{Z} \to i_*\mathbb{Z} \to 0$$

induces an isomorphism $H_c^{n+1}(Y \setminus X, \mathbb{Z}) \cong H^{n+1}(Y, X; \mathbb{Z})$, and likewise $H_c^{n+2r+1}(P \setminus \mathcal{X}, \mathbb{Z}) \cong H^{n+2r+1}(P, \mathcal{X}; \mathbb{Z})$. To conclude the proof, it suffices to show that $H^n(\mathcal{X},\mathbb{Z})_v \cong H^{n+1}(Y, X)$ and $H^{n+2r}(\mathcal{X},\mathbb{Z})_v \cong H^{n+2r+1}(P, \mathcal{X})$, i.e., we should verify that the restriction maps $i^* : H^{n+1}(Y,\mathbb{Z}) \to H^{n+1}(X,\mathbb{Z})$ and $\iota^* : H^{n+2r+1}(P,\mathbb{Z}) \to H^{n+2r+1}(\mathcal{X},\mathbb{Z})$ are injective. To show the injectivity of i^*, consider the commutative diagram

$$\begin{array}{ccc} H^{n-1}(Y,\mathbb{Z}) & \xrightarrow{L_Y} & H^{n+1}(Y,\mathbb{Z}) \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\ H^{n-1}(X,\mathbb{Z}) & \xrightarrow{L_X} & H^{n+1}(X,\mathbb{Z}). \end{array}$$

The Lefschetz hyperplane theorem implies that $H^{n-1}(Y,\mathbb{Z}) \cong H^{n-1}(X,\mathbb{Z})$, hence $H^{n-1}(X,\mathbb{Z})$ is torsion free. As $L_X : H^{n-1}(X,\mathbb{Q}) \to H^{n+1}(X,\mathbb{Q})$ is an isomorphism by the hard Lefschetz theorem, we find that L_X is injective on cohomology with $\mathbb{Z}$–coefficients. Hence i^* is injective if L_Y is an isomorphism.

For the second statement, note that the Leray spectral sequence for the map φ degenerates at E_2; cf. [41, Thm. 1.2]. This yields isomorphisms

$$\begin{aligned} H^{n+2r-1}(P,\mathbb{Z}) &\cong H^{n+2r-1}(Y,\mathbb{Z}) \oplus \ldots \oplus H^{n-1}(Y,\mathbb{Z})(-r) \\ H^{n+2r+1}(P,\mathbb{Z}) &\cong H^{n+2r+1}(Y,\mathbb{Z}) \oplus \ldots \oplus H^{n+1}(Y,\mathbb{Z})(-r), \end{aligned}$$

hence the groups $H^{n+2r-1}(P,\mathbb{Z})$ and $H^{n+2r+1}(P,\mathbb{Z})$ are torsion free. Combining this assertion with the hard Lefschetz theorem, we find that the Lefschetz operator $L_P : H^{n+2r-1}(P,\mathbb{Z}) \to H^{n+2r+1}(P,\mathbb{Z})$ is injective. Since $b_{n-1}(Y) = b_{n+1}(Y)$ by hypothesis, we obtain

$$\begin{aligned} b_{n+2r+1}(P) - b_{n+2r-1}(P) &= b_{n+2r+1}(Y) - b_{n-1}(Y) \\ &= b_{n+2r+1}(Y) - b_{n+1}(Y) = 0 \end{aligned}$$

where the last equality follows from hypothesis (i). Hence $\mathrm{im}(L_P) \subset H^{n+2r+1}(P,\mathbb{Z})$ is a subgroup of finite index, d say. Let $\iota : \mathcal{X} \hookrightarrow P$ be the inclusion map, and consider the commutative diagram

$$\begin{array}{ccc} H^{n+2r-1}(P,\mathbb{Z}) & \xrightarrow{L_P} & H^{n+2r+1}(P,\mathbb{Z}) \\ \downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle \iota^*} \\ H^{n+2r-1}(\mathcal{X},\mathbb{Z}) & \xrightarrow{L_{\mathcal{X}}} & H^{n+2r+1}(\mathcal{X},\mathbb{Z}). \end{array}$$

Given $\alpha \in H^{n+2r+1}(P,\mathbb{Z})$, we have $d\alpha \in \mathrm{im}(L_P)$. As $L_{\mathcal{X}}$ is injective, the above diagram shows that $\iota^*|_{\mathrm{im}(L_P)}$ is injective. This implies the injectivity of $\iota^* : H^{n+2r+1}(P,\mathbb{Z}) \to H^{n+2r+1}(\mathcal{X},\mathbb{Z})$, since $H^{n+2r+1}(P,\mathbb{Z})$ is torsion free. $\square$

2.3 Leray spectral sequence

Consider a quadric bundle $\mathcal{X} \to S$ that is defined by a quadratic form $q \in H^0(S, S^2E^\vee \otimes L)$.The quadratic form q induces a section $\overline{q} : S \to \mathbb{P}(S^2E^\vee)$. In the sequel we identify S with its image $\overline{q}(S) \subset \mathbb{P}(S^2E^\vee)$. The tautological exact sequence

$$0 \to \xi^{-1} \to p^*S^2E^\vee \to Q \to 0$$

on $\mathbb{P}(S^2E^\vee)$ induces a 'universal section' $q_{\mathrm{univ}} \in H^0(P, p^*S^2E^\vee \otimes \xi)$. Put

$$\Delta_i = \Delta_i(q) = \{s \in S \mid \mathrm{corank}\,(q(s)) \geq i\}.$$

We have

$$q_{\mathrm{univ}}|_S = q, \quad \Delta_i(q_{\mathrm{univ}}) \cap S = \Delta_i(q);$$

cf. [67, p. 415 (12)].

Definition 2.3.1 *The quadric bundle $\mathcal{X}$ is called* regular *if $\Delta_i(q_{\mathrm{univ}})$ intersects S transversally for all $i \geq 1$.*

Proposition 2.3.2 *Suppose that $\mathcal{X} \to S$ is a regular quadric bundle. Then*

(i) $\Delta_i(q)$ *is irreducible;*

(ii) $\mathrm{codim}\, \Delta_i(q) = \binom{i+1}{2}$ *if Δ_i is nonempty;*

(iii) $\mathrm{Sing}(\Delta_i(q)) = \Delta_{i+1}(q)$.

Proof: See [67, §2]. $\square$

Proposition 2.3.3 *Let $f : \mathcal{X} \to S$ be a regular quadric bundle. The sheaves $R^q f_* \mathbb{Z}$ are constructible with respect to the stratification $\{\Delta_i\}$.*

Proof: We define

$$\begin{aligned} U_\alpha &= \Delta_\alpha \setminus \Delta_{\alpha+1}, \quad \mathcal{X}_\alpha = \mathcal{X} \times_S U_\alpha \\ \Sigma_\alpha &= \{x \in \mathcal{X}_\alpha \mid df : T_x\mathcal{X}_\alpha \to T_{f(x)}S \text{ is not surjective}\}. \end{aligned}$$

The projection $f_\alpha = f|_{\Sigma_\alpha} : \Sigma_\alpha \to U_\alpha$ is a $\mathbb{P}^{\alpha-1}$–bundle. Write

$$V_\alpha = \mathcal{X}_\alpha \setminus \Sigma_\alpha, \quad g_\alpha : V_\alpha \to U_\alpha, \quad i_\alpha : \Sigma_\alpha \hookrightarrow \mathcal{X}_\alpha, \quad j_\alpha : V_\alpha \hookrightarrow \mathcal{X}_\alpha.$$

Applying the functor Rf_* to the exact sequence

$$0 \to j_{\alpha,!}\mathbb{Z} \to \mathbb{Z} \to i_{\alpha,*}\mathbb{Z} \to 0$$

and taking cohomology, we find an exact sequence

$$R^q g_{\alpha,!}\mathbb{Z} \to R^q f_*\mathbb{Z}|_{U_\alpha} \to R^q f_{\alpha,*}\mathbb{Z}.$$

Put $n = \dim_S(\mathcal{X})$. Proposition 2.2.1 shows that

$$R^q f_*\mathbb{Z}|_{U_\alpha} = \begin{cases} R^q f_{\alpha,*}\mathbb{Z} & 0 \le q \le 2s-2 \\ 0 & q = 2s-1 \\ R^q g_{\alpha,!}\mathbb{Z} & 2s \le q \le 2n. \end{cases}$$

Since f_α and g_α are smooth morphisms, the sheaves $R^q f_{\alpha,*}\mathbb{Z}$ and $R^q g_{\alpha,!}\mathbb{Z}$ are locally constant. $\square$

Let $\varphi : \mathbb{P}(E^\vee) \to S$ be the projection map. By analogy with Definition 2.2.3 we put

$$\begin{aligned} (R^k f_*\mathbb{Z})_0 &= \ker(i_* : R^k f_*\mathbb{Z} \to R^{k+2}\varphi_*\mathbb{Z}) \\ (R^k f_*\mathbb{Z})_v &= \operatorname{coker}(i^* : R^k\varphi_*\mathbb{Z} \to R^k f_*\mathbb{Z}). \end{aligned}$$

Proposition 2.3.4 *Let $s \ge 0$ be an integer such that $n+s$ is even. Consider the inclusion maps*

$$j_s : U_s \to S, \quad j : S \setminus \Delta_s \to S, \quad i_s : \Delta_s \to S.$$

(i) $(R^q f_*\mathbb{Z})_0 = (R^q f_*\mathbb{Z})_v = 0$ if $q < n$;

(ii) There exists a rank one local system $\mathbb{V}_{n+s}$ on U_s that fits into an exact sequence

$$0 \to j_!\mathbb{Z}/2 \to (R^{n+s} f_*\mathbb{Z})_v \to j_{s,!}\mathbb{V}_{n+s} \to 0.$$

Proof: Part (i) follows from Proposition 2.2.6 and the proper base change theorem. For the second part, consider the exact sequence

$$0 \to j_!j^*(R^{n+s}f_*\mathbb{Z})_v \to (R^{n+s}f_*\mathbb{Z})_v \to i_{s,*}i_s^*(R^{n+s}f_*\mathbb{Z})_v \to 0$$

and use Propositions 2.2.6 and 2.3.3 to rewrite the terms on the left hand and right hand side. $\square$

For the statement of the next result we need the following terminology. A mixed Hodge structure $(H, F^\bullet, W_\bullet)$ has weights $\leq k$ (resp. $\geq k$) if $\mathrm{Gr}_q^W H = 0$ for all $q > k$ (resp. for all $q < k$).

Theorem 2.3.5 *Let $f : X \to S$ be a projective morphism of projective varieties. The Leray spectral sequence*

$$E_2^{p,q}(f) = H^p(S, R^q f_*\mathbb{Z}) \Rightarrow H^{p+q}(X, \mathbb{Z})$$

lifts to the category of mixed Hodge structures (MHS), i.e., the E_r terms carry a MHS that is compatible with the differentials d_r and the induced MHS on the E_∞ terms is compatible with the MHS on $H^*(X)$. Furthermore, the mixed Hodge structure on $E_r^{p,q}(f)$ has weights $\leq p + q$.

Proof: These results follow from M. Saito's theory of mixed Hodge modules [61], [62]. Recently, Arapura found a direct proof; see [3, Cor. 3.2] and [3, Thm 4.5]. $\square$

Proposition 2.3.6 *Suppose that $H^*(S, \mathbb{Z})$ is torsion free. Then there exists a spectral sequence $E_r^{p,q}(f)_v \Rightarrow H^{p+q}(\mathcal{X}, \mathbb{Z})_v$ with $E_2^{p,q}(f)_v \cong H^p(S, (R^q f_*\mathbb{Z})_v)$.*

Proof: The inclusion $i : \mathcal{X} \to \mathbb{P}(E^\vee)$ induces a homomorphism of Leray spectral sequences $i^* : E_r^{p,q}(\varphi) \to E_r^{p,q}(f)$. Let $E_r^{p,q}(f)_v$ be the cokernel of this map. Since the Leray spectral sequence for $\varphi : \mathbb{P}(E^\vee) \to S$ degenerates at E_2, the maps $d_r(f)$ vanish on the image of i^* and induce maps

$$\overline{d}_r : E_r^{p,q}(f)_v \to E_r^{p+r,q-r+1}(f)_v.$$

To show that the terms $\{E_r^{p,q}(f)_v, \overline{d}_r\}$ form a spectral sequence we have to check that the cohomology at the middle term of the complex

$$\mathcal{E}_r^\bullet(f)_v = (\to E_r^{p-r,q+r-1}(f)_v \xrightarrow{\overline{d}_r} E_r^{p,q}(f)_v \xrightarrow{\overline{d}_r} E_r^{p+r,q-r+1}(f)_v \to) \qquad (2.2)$$

is isomorphic to $E_{r+1}^{p,q}(f)_v$ for all $r \geq 2$.

For $r = 2$ we consider the long exact sequence

$$\to E_2^{p-1,q}(f)_v \xrightarrow{\delta} E_2^{p,q}(\varphi) \xrightarrow{i^*} E_2^{p,q}(f)_v \xrightarrow{\delta}$$

induced by the exact sequence of sheaves

$$0 \to R^q\varphi_*\mathbb{Z} \to R^q f_*\mathbb{Z} \to (R^q f_*\mathbb{Z})_v \to 0.$$

By Theorem 2.3.5 the terms $E_2^{p,q}(f) \otimes \mathbb{Q}$ carry a MHS with weights $\leq p+q$, hence the induced MHS on $E_2^{p,q}(f)_v \otimes \mathbb{Q}$ also has weights $\leq p + q$. Since

$E_2^{p,q}(\varphi) \otimes \mathbb{Q}$ carries a pure Hodge structure of weight $p+q$, the connecting homomorphism

$$\delta \otimes \mathrm{id} : E_2^{p-1,q}(f)_v \otimes \mathbb{Q} \to E_2^{p,q}(\varphi) \otimes \mathbb{Q}$$

is zero for all (p,q). Since $E_2^{p,q}(\varphi) \cong H^p(S,\mathbb{Z})$ is torsion free, this implies that $\delta = 0$.

For higher values of r we consider the complex $\mathcal{E}_r^\bullet(f)_v$ as in (2.2) where the term $E_r^{p,q}(f)_v$ is placed in degree $p+q$. Similarly we define the complexes $\mathcal{E}_r^\bullet(f)$, $\mathcal{E}_r^\bullet(\varphi)$. The previous argument shows that we have an exact sequence of complexes

$$0 \to \mathcal{E}_2^\bullet(\varphi) \to \mathcal{E}_2^\bullet(f) \to \mathcal{E}_2^\bullet(f)_v \to 0.$$

As before, the maps

$$\delta : H^{p+q-1}(\mathcal{E}_2^\bullet(f)_v) \to H^{p+q}(\mathcal{E}_2^\bullet(\varphi))$$

are zero by mixed Hodge theory; hence we obtain an exact sequence

$$\begin{array}{ccccccc} 0 \to & H^{p+q}(\mathcal{E}_2^\bullet(\varphi)) & \to & H^{p+q}(\mathcal{E}_2^\bullet(f)) & \to & H^{p+q}(\mathcal{E}_2^\bullet(f)_v) & \to 0 \\ & \| & & \| & & & \\ & E_3^{p,q}(\varphi) & & E_3^{p,q}(f) & & & \end{array}$$

that induces an isomorphism $H^{p+q}(\mathcal{E}_2^\bullet(f)_v) \cong E_3^{p,q}(f)_v$. Continuing this process inductively, we obtain exact sequences

$$0 \to \mathcal{E}_r^\bullet(\varphi) \to \mathcal{E}_r^\bullet(f) \to \mathcal{E}_r^\bullet(f)_v \to 0$$

and isomorphisms $H^{p+q}(\mathcal{E}_r^\bullet(f)_v) \cong E_{r+1}^{p,q}(f)_v$ for all $r \geq 2$. Since the spectral sequence $E_r^{p,q}(f)$ degenerates after a finite number of steps, the same holds for $E_r^{p,q}(f)_v$ and we obtain an isomorphism

$$E_\infty^{p,q}(f)_v \cong \mathrm{coker}\,(E_\infty^{p,q}(\varphi) \to E_\infty^{p,q}(f)).$$

□

2.4 Local systems

To study the spectral sequence $E_r^{p,q}(f)_v$ introduced in the previous section, we recall some of the known vanishing theorems for cohomology with values in a local system.

Proposition 2.4.1 *Let U be a smooth affine variety of dimension n, and let $\mathbb{L}$ be a local system of abelian groups on U. Then* *(i)* $H^q(U, \mathbb{L}) = H_q(U, \mathbb{L}) = 0$ *for all* $q > n$;

(ii) $H^q_c(U, \mathbb{L}) = 0$ *for all* $q < n$.

Proof: As U is affine, it has the homotopy type of a CW–complex with cells in real dimension $\leq n$; cf. [2], [47]. This implies the first statement. The second statement then follows from the isomorphism

$$H_k(U, \mathbb{L}) \cong H_c^{2n-k}(U, \mathbb{L}^\vee)$$

which is obtained by Poincaré–Verdier duality. □

Remark 2.4.2 M. Artin proved a more general version of this statement for constructible sheaves that is valid in arbitrary characteristic [5, Cor. 3.2]. In characteristic zero Nori's "basic lemma" [57] gives an algebraic proof.

Corollary 2.4.3 (Barth–Lefschetz for double coverings) *Let X and S be smooth, projective varieties and let $\pi : X \to S$ be a double covering such that the branch locus $\Delta \subset S$ is an ample divisor. Then the map*

$$\pi^* : H^k(S, \mathbb{Z}) \to H^k(X, \mathbb{Z})$$

is an isomorphism for all $k < n$. The map π^* is injective for $k = n$.

Proof: Let $(\pi_*\mathbb{Z})_v$ be the cokernel of the injective map $\mathbb{Z}_S \to \pi_*\mathbb{Z}$. There exists a rank one local system $\mathbb{L}$ on $U = S \setminus \Delta$ such that $(\pi_*\mathbb{Z})_v \cong j_!\mathbb{L}$, where $j : U \to S$ is the inclusion map. The statement then follows from Proposition 2.4.1 using the exact sequence

$$\begin{array}{ccccc} H^{k-1}(S, (\pi * \mathbb{Z})_v) & \to H^k(S, \mathbb{Z}) \xrightarrow{\pi^*} H^k(X, \mathbb{Z}) \to & H^k(S, (\pi_*\mathbb{Z})_v) \\ \| & & \| \\ H_c^{k-1}(U, \mathbb{L}) & & H_c^k(U, \mathbb{L}). \end{array}$$

□

Remark 2.4.4 If $S = \mathbb{P}^r$ the statement follows from [44, Cor. 3.2]. A more general statement, valid for cyclic coverings of arbitrary degree, can be found in [21]. Note that both authors prove a stronger result, valid for homotopy groups.

The following generalisation of Proposition 2.4.1 is due to Esnault and Viehweg.

Proposition 2.4.5 *Let S be a smooth projective variety of dimension n, and let $\mathcal{A} = \{D_1, \ldots, D_k\}$ be an arrangement of hypersurfaces in S such that $D = \cup_{i=1}^k D_i$ is a divisor with strict normal crossings. Let $\mathbb{L}$ be a rank one local system of $\mathbb{Q}$–vector spaces on $M(\mathcal{A}) = S \setminus D$. If $M(\mathcal{A})$ is affine and the monodromy of $\mathbb{L}$ around D_i is nontrivial for all i, then $H^q(M(\mathcal{A}), \mathbb{L}) = H^q_c(M(\mathcal{A}), \mathbb{L}) = 0$ for all $q \neq n$.*

Proof: See [33, Cor. (1.5)]; cf. also [46, Thm. 3.2] or [31, Thm. 3.4.4]. □

Remark 2.4.6 Esnault and Viehweg prove a more general result, valid for local systems of arbitrary rank. For our purposes the statement for rank one local systems suffices.

Chapter 3

The action of the correspondence

In this chapter we study the cohomology of regular quadric bundles using the action of a correspondence defined by a family of linear subspaces on the quadrics. The methods used depend on the parity of the relative dimension of the quadric bundle.

3.1 Even–dimensional quadrics

Let $Q \subset \mathbb{P}^{2m+1}$ be a smooth quadric. The Fano scheme $F_m(Q)$ parametrising m–dimensional linear subspaces contained in Q has two connected components F', F'' that are both isomorphic to a homogeneous space F of dimension $d = \frac{1}{2}m(m+1)$, the so–called *spinor variety.* If $Q \subset \mathbb{P}^{2m+1}$ is a quadric of corank one, $F_m(Q)$ has one connected component, isomorphic to F. These classical results go back to E. Cartan; cf. [35, p. 390] for a modern treatment.

Let $f : \mathcal{X} \to S$ be a regular quadric bundle, and let $F_m(\mathcal{X}/S)$ be the associated relative Fano scheme [1]. Let

$$F_m(\mathcal{X}/S) \xrightarrow{h} W \xrightarrow{\pi} S$$

be the Stein factorisation of the structure morphism $g : F_m(\mathcal{X}/S) \to S$. The morphism $\pi : W \to S$ is a double covering that is ramified over the discriminant locus $\Delta = \Delta_1$. Note that

$$W \text{ is singular} \iff \Delta_2 \neq \emptyset \iff \dim S \geq 3.$$

Let $p : \Gamma \to F_m(\mathcal{X}/S)$ be the universal family of m–planes. Note that Γ can be seen as a correspondence, more precisely

$$\Gamma = \{(s, x, \Lambda) \mid x \in \Lambda \subset Q_s\}$$

is the incidence correspondence in the fibered product $F_m(\mathcal{X}/S) \times_S \mathcal{X}$.

The following definition is taken from [37, 1.1].

Definition 3.1.1 *Let S be a smooth quasi–projective variety, and let X and Y be projective schemes over S. The group*

$$\mathrm{Corr}^p_S(X,Y) = \mathrm{CH}_{\dim Y - p}(X \times_S Y)$$

is called the group of relative correspondences of degree p from X to Y.

Remark 3.1.2 The Chow group $\mathrm{CH}_*(X \times_S Y)$ is defined as in [34]. There is a natural map

$$\mathrm{Corr}^p_S(X,Y) = \mathrm{CH}_{\dim Y-p}(X \times_S Y) \to \mathrm{CH}_{\dim Y-p}(X \times Y) = \mathrm{Corr}^p(X,Y)$$

that sends a relative correspondence to the associated absolute correspondence.

In the above definition one usually assumes that X and Y are smooth. The following lemma shows that one only needs smoothness of Y to define the action of (relative) correspondences.

Lemma 3.1.3 (Corti–Hanamura) *Let*

$$\begin{array}{ccc} X \times_S Y & \to & Y \\ \downarrow & & \downarrow g \\ X & \xrightarrow{f} & S \end{array}$$

be a commutative diagram of schemes such that f is proper and Y is smooth. Then

$$H^{BM}_{2\dim Y+i-j}(X \times_S Y, \mathbb{Z}) \cong \mathrm{Hom}_{D^b_c(\mathbb{Z}_S)}(Rf_*\mathbb{Z}[i], Rg_*\mathbb{Z}[j])$$

Proof: The result is proved in [22, Lemma 2.21 (2)] for $\mathbb{Q}$–coefficients. The same proof goes through for $\mathbb{Z}$–coefficients. $\square$

Corollary 3.1.4 *Under the above hypotheses, there exist natural maps*

$$\begin{array}{rcl} \mathrm{Corr}^p_S(X,Y) & \to & \mathrm{Hom}_D(Rf_*\mathbb{Z}, Rg_*\mathbb{Z}[2p]) \\ \mathrm{Corr}^p_S(X,Y) & \to & \mathrm{Hom}(H^k(X,\mathbb{Z}), H^{k+2p}(Y,\mathbb{Z})) \end{array}$$

for all $k \geq 0$.

Proof: The first map is the composition of the cycle class map

$$\mathrm{cl} : \mathrm{CH}_{\dim Y - p}(X \times_S Y) \to H^{BM}_{2\dim Y - 2p}(X \times_S Y)$$

to Borel–Moore homology and the isomorphism of the previous lemma. Applying this construction with $S = \operatorname{Spec} \mathbb{C}$ gives a map

$$\mathrm{Corr}^p(X, Y) \to \mathrm{Hom}(H^k(X, \mathbb{Z}), H^{k+2p}(Y, \mathbb{Z}));$$

composition with the natural map $\mathrm{Corr}^p_S(X, Y) \to \mathrm{Corr}^p(X, Y)$ gives the second map. $\square$

Consider the commutative diagram

$$\begin{tikzcd}
\Gamma \arrow[r, "q"] \arrow[d, "p"] & \mathcal{X} \arrow[d, "f"] \\
F_m(\mathcal{X}/S) \arrow[r, "g"] & S.
\end{tikzcd}$$

There exists a subvariety $W' \subset F_m(\mathcal{X}/S)$ such that $h' = h|_{W'}$ is a finite morphism. Put

$$\Gamma' = \Gamma \times_{F_m(\mathcal{X}/S)} W'.$$

Corollary 3.1.5 *The correspondence Γ' induces a homomorphism of constructible sheaves*

$$\Gamma'_* : \pi_* \mathbb{Z} \to R^{2m} f_* \mathbb{Z}.$$

If $\dim S \leq 2$*, there exists a homomorphism*

$${}^t\Gamma'_* : R^{2m} f_* \mathbb{Z} \to \pi_* \mathbb{Z}.$$

Proof: For the first statement, apply Corollary 3.1.4 to the relative correspondence $\Gamma' \in \mathrm{Corr}^m_S(W, \mathcal{X})$. If $\dim S \leq 2$ we can apply the Corollary to ${}^t\Gamma'$ to obtain a homomorphism in the opposite direction. $\square$

Consider the following commutative diagram

$$\begin{tikzcd}
\Gamma' \arrow[r] \arrow[d] & \Gamma \arrow[r, "q"] \arrow[d, "p"] & \mathcal{X} \arrow[r, "i"] \arrow[dd, "f"] & \mathbb{P}(S^2 E^\vee) \arrow[ddl, "\varphi"] \\
W' \arrow[r, "i'"] \arrow[dr, "h'"'] & F_m(\mathcal{X}/S) \arrow[d, "h"] \arrow[dr, "g"] & & \\
& W \arrow[r, "\pi"] & S. &
\end{tikzcd}$$

The variable and primitive part of the sheaf $\pi_*\mathbb{Z}$ are defined by

$$(\pi_*\mathbb{Z})_v = \mathrm{coker}\,(\mathbb{Z}_S \to \pi_*\mathbb{Z}_W), \quad (\pi_*\mathbb{Z})_0 = \ker(\pi_*\mathbb{Z}_W \xrightarrow{Tr} \mathbb{Z}_S).$$

Lemma 3.1.6 *The correspondence Γ' induces homomorphisms*

$$\begin{aligned} \Gamma'_* &: (\pi_*\mathbb{Z})_v \to (R^{2m} f_*\mathbb{Z})_v \\ \Gamma'_* &: H^k(W,\mathbb{Z}) \to H^{2m+k}(\mathcal{X},\mathbb{Z})_v \end{aligned}$$

for all $k \geq 0$. If $\dim S \leq 2$ *there exist homomorphisms*

$$\begin{aligned} {}^t\Gamma'_* &: (R^{2m} f_*\mathbb{Z})_0 \to (\pi_*\mathbb{Z})_0 \\ {}^t\Gamma'_* &: H^{2m+k}(\mathcal{X},\mathbb{Z})_0 \to H^k(W,\mathbb{Z})_0. \end{aligned}$$

Proof: Put

$$p'' = \pi\circ p', \quad q'' = i\circ q', \quad \Gamma'' = q''_*\circ(p'')^*.$$

We have

$$\Gamma''_* = i_*\circ q'_*\circ(p')^*\circ\pi^* = i_*\Gamma'_*\circ\pi^*.$$

Apply the functor i^* and use the identity $i^*i_* = \mathrm{id}$ [11, (1.4.1.2)] to obtain

$$i^*\circ\Gamma''_* = \Gamma'_*\circ\pi^*.$$

This formula shows that $\Gamma'_*(\mathrm{im}\,\pi^*) \subseteq \mathrm{im}\,i^*$, hence Γ'_* maps the primitive quotient of $\pi_*\mathbb{Z}$ (resp. $H^k(W,\mathbb{Z})$) to the primitive quotient of $R^{2m} f_*\mathbb{Z}$ (resp. $H^{2m+k}(\mathcal{X},\mathbb{Z})$). The second assertion follows from the equality

$${}^t\Gamma''_*\circ i_* = \pi_* {}^t\Gamma'_*.$$

□

Lemma 3.1.7 *The map $\Gamma'_* : (\pi_*\mathbb{Z})_v \to (R^{2m} f_*\mathbb{Z})_v$ is an isomorphism.*

Proof: Given $u \in U$, the induced map on the fibers

$$\Gamma'_{u,*} : H^0(\pi^{-1}(u),\mathbb{Z}) \to H^{2m}(f^{-1}(u),\mathbb{Z})$$

sends the basis $\{[u'],[u'']\}$ to the basis $\{[\Lambda(u')],[\Lambda(u'')]\}$. Hence we obtain an isomorphism

$$H^0(\pi^{-1}(u),\mathbb{Z})_v \xrightarrow{(\Gamma'_u)_*} H^{2m}(f^{-1}(u),\mathbb{Z})_v$$

for all $u \in U$. As the sheaves $(\pi_*\mathbb{Z})_v$ and $(R^{2m} f_*\mathbb{Z})_v$ are zero outside U, this implies that

$$(\pi_*\mathbb{Z})_v \xrightarrow{\Gamma'_*} (R^{2m} f_*\mathbb{Z})_v$$

is an isomorphism. □

Lemma 3.1.8 *If $H^{k+1}(S,\mathbb{Z})$ has no 2–torsion then $H^k(S,(\pi_*\mathbb{Z})_v) \cong H^k(W,\mathbb{Z})_v$.*

Proof: The exact sequence

$$0 \to \mathbb{Z}_S \to \pi_*\mathbb{Z}_W \to (\pi_*\mathbb{Z})_v \to 0$$

induces a long exact sequence

$$H^k(S,\mathbb{Z}) \xrightarrow{\pi^*} H^k(W,\mathbb{Z}) \to H^k(S,(\pi_*\mathbb{Z})_v) \xrightarrow{\delta} H^{k+1}(S,\mathbb{Z}) \xrightarrow{\pi^*} H^{k+1}(W,\mathbb{Z}).$$

As $\pi_*\circ\pi^* = 2.id$, the kernel of π^* is a 2–torsion group. Hence the hypothesis implies that $H^k(S,(\pi_*\mathbb{Z})_v) \cong H^k(W,\mathbb{Z})_v$. $\square$

Corollary 3.1.9 *Suppose that $H^*(S,\mathbb{Z})$ is torsion free, and consider the homomorphism*

$$\Gamma'_* : H^r(W,\mathbb{Z})_v \to H^{2m+r}(\mathcal{X},\mathbb{Z})_v.$$

(i) Γ'_* is surjective if and only if $E_\infty^{r-2k,2m+2k}(f)_v = 0$ for all $k > 0$;

(ii) Γ'_* is injective if and only if $E_2^{r,2m}(f)_v = E_\infty^{r,2m}(f)_v$.

Proof: The homomorphism $\Gamma'_* : \pi_*\mathbb{Z} \to R^{2m}f_*\mathbb{Z}$ induces a homomorphism between the Leray spectral sequences of the maps π and f. By Lemma 3.1.7 we have $H^r(W,\mathbb{Z})_v = H^r(S,(\pi_*\mathbb{Z})_v)$, and by Proposition 2.3.6 there exists a spectral sequence $E_r^{p,q}(f)_v \Rightarrow H^{p+q}(\mathcal{X},\mathbb{Z})_v$. The homomorphism Γ'_* induces an isomorphism

$$H^r(W,\mathbb{Z})_v = H^r(S,(\pi_*\mathbb{Z})_v) \xrightarrow{\sim} H^r(S,(R^{2m}f_*\mathbb{Z})_v) = E_2^{r,2m}(f)_v.$$

Since $\Gamma'_* : H^r(W,\mathbb{Z})_v \to H^{2m+r}(\mathcal{X},\mathbb{Z})_v$ is compatible with the Leray spectral sequences, its image is contained in $E_\infty^{r,2m}(f)_v$. Proposition 2.3.4 shows that the terms $E_2^{p,q}(f)_v$ vanish if $q \neq 2m+2k$, $0 \le k \le m$. Hence there is a natural injective map $E_\infty^{r,2m}(f)_v \hookrightarrow E_\infty^{2m+r}(f)_v$ and there are no outgoing differentials at position $(r,2m)$, so we obtain a surjective map $E_2^{r,2m}(f)_v \to E_\infty^{r,2m}(f)_v$. The map $\Gamma'_* : H^r(W,\mathbb{Z})_v \to H^{2m+r}(\mathcal{X},\mathbb{Z})_v$ then factors in the following way.

$$\begin{tikzcd}
H^r(W,\mathbb{Z})_v \arrow[d, "\cong"] \arrow[ddrr, "\Gamma'_*"] & & \\
H^r(S,(\pi_*\mathbb{Z})_v) \arrow[d, "\Gamma'_*"] & & \\
E_2^{r,2m}(f)_v \arrow[r] & E_\infty^{r,2m}(f)_v \arrow[r] & H^{2m+r}(\mathcal{X},\mathbb{Z})_v.
\end{tikzcd}$$

It follows that

$$\operatorname{im}(\Gamma'_*) \cong E_\infty^{r,2m}(f)_v, \quad ker(\Gamma'_*) \cong \ker(E_2^{r,2m}(f)_v \to E_\infty^{r,2m}(f)_v).$$

Combining this with the (non–canonical) isomorphism

$$H^{2m+r}(\mathcal{X}, \mathbb{Z})_v \cong \bigoplus_{0 \leq 2k \leq r} E_\infty^{r-2k, 2m+2k}(f)_v$$

we obtain the statements (i) and (ii). $\square$

Corollary 3.1.10 *The homomorphism $\Gamma'_* : H^r(W, \mathbb{Q})_v \to H^{2m+r}(\mathcal{X}, \mathbb{Q})_v$ is injective if and only if $H^r(W, \mathbb{Q})_v$ carries a pure Hodge structure of weight r.*

Proof: As we have seen in Chapter 2, Γ'_* is a morphism of mixed Hodge structures of type (m, m). If Γ'_* is injective, the resulting inclusion $H^r(W, \mathbb{Q})_v \subseteq H^{2m+r}(\mathcal{X}, \mathbb{Q})_v(m)$ shows that $H^r(W, \mathbb{Q})_v$ carries a pure Hodge structure of weight r. Conversely, if $H^r(W, \mathbb{Q})_v$ carries a pure Hodge structure then Γ'_* is injective. Indeed, condition (ii) of the previous Corollary is satisfied if and only if the maps

$$d_{2k+1} : E_{2k+1}^{r-2k-1, 2m+2k}(f)_v \to E_{2k+1}^{r,2m}(f)_v$$

are zero for all $k > 0$. By Theorem 2.3.5 the terms on the left hand side carry a MHS with weights $\leq 2m + r - 1$, and the isomorphism of mixed Hodge structures $E_2^{r,2m}(f)_v \otimes \mathbb{Q} \cong H^r(W, \mathbb{Q})_v(-m)$ implies that the right hand side carries a pure Hodge structure of weight $2m + r$. Hence $d_{2k+1} = 0$ for all $k > 0$. $\square$

Theorem 3.1.11 *Let $f : \mathcal{X} \to S$ be a regular quadric bundle. Put $r = \dim S$. If $r \leq 2$ then*

$$\Gamma'_* : H^r(W, \mathbb{Q})_v \to H^{2m+r}(\mathcal{X}, \mathbb{Q})_v$$

is an isomorphism. If $r = 3$ then Γ'_* is surjective.

Proof: If $r \leq 2$, Proposition 2.3.4 implies that $E_2^{p,q}(f)_v \otimes \mathbb{Q} = 0$ if $q \neq 2m$. Hence the only term that contributes to $H^{2m+r}(\mathcal{X}, \mathbb{Q})_v$ is $E_2^{r,2m}(f)_v \otimes \mathbb{Q} = E_\infty^{r,2m}(f)_v \otimes \mathbb{Q}$, and Γ'_* is an isomorphism by Corollary 3.1.9.

For $r = 3$ Proposition 2.3.4 implies that

$$H^{2m+3}(\mathcal{X}, \mathbb{Q})_v \cong E_\infty^{3,2m}(f)_v \oplus E_\infty^{1,2m+2}(f)_v.$$

As $\mathcal{X}$ is regular, Δ_2 is a finite set of points. Hence $E_2^{1,2m+2}(f)_v = H^1(\Delta_2, \mathbb{V}_{2m+2}) = 0$, and Corollary 3.1.9 (i) shows that Γ'_* is surjective. $\square$

For integer coefficients we have the following results.

Theorem 3.1.12 *Let $f : \mathcal{X} \to S$ be a regular quadric bundle. If $r = \dim S < 2$ then*

$$\Gamma'_* : H^r(W, \mathbb{Z})_v \to H^{2m+r}(\mathcal{X}, \mathbb{Z})_v$$

is an isomorphism.

Proof: The proof of Theorem 3.1.11 goes through if $r < 2$, since the cohomology of the sheaves $(R^q f_* \mathbb{Z})_v$ with $q > 2m$ does not enter the argument. Indeed, if $r < 2$ then $E_2^{r-2k,2m+2k}(f)_v = 0$ for all $k \geq 1$, and we obtain $H^{2m+r}(\mathcal{X}, \mathbb{Z})_v = E_\infty^{r,2m}(f)_v$. Since $E_2^{r-2k-1,2m+2k}(f)_v = 0$ there are no incoming differentials, hence $E_\infty^{r,2m}(f)_v = E_2^{r,2m}(f)_v$. Note that if $r < 2$ the hypothesis of Corollary 3.1.9 is satisfied. $\square$

Theorem 3.1.13 *Let S be a smooth projective surface, and let $f : \mathcal{X} \to S$ be a regular quadric bundle. Suppose that*

(i) $H_1(S, \mathbb{Z}) = 0$;

(ii) The discriminant locus $\Delta \subset S$ is an ample divisor.

Then we have an exact sequence

$$0 \to H^2(W, \mathbb{Z})_v \xrightarrow{\Gamma'_*} H^{2m+2}(\mathcal{X}, \mathbb{Z})_v \to \mathbb{Z}/2 \to 0.$$

Proof: Since $\mathcal{X} \to S$ is regular, Proposition 2.3.4 implies that $E_\infty^{2m+2}(f)_v \cong E_\infty^{2,2m}(f)_v \oplus E_\infty^{0,2m+2}(f)_v$. As in the proof of Theorem 3.1.11, inspection of the spectral sequence shows that $E_\infty^{2,2m}(f)_v \cong E_2^{2,2m}(f)_v$. Hence we obtain a short exact sequence

$$\begin{array}{ccccccc} 0 \to & E_\infty^{2,2m}(f)_v & \to & E_\infty^{2m+2}(f)_v & \to & E_\infty^{0,2m+2}(f)_v & \to 0 \\ & \| & & \| & & & \\ & H^2(W, \mathbb{Z})_v & \xrightarrow{\Gamma'_*} & H^{2m+2}(\mathcal{X}, \mathbb{Z})_v. & & & \end{array}$$

It remains to show that $E_\infty^{0,2m+2}(f)_v \cong \mathbb{Z}/2$. As $(R^{2m+2} f_* \mathbb{Z})_v \cong \mathbb{Z}/2$ by Proposition 2.3.4, we have $E_2^{0,2m+2}(f)_v \cong \mathbb{Z}/2$. There are no incoming differentials at position $(0, 2m)$, and the only relevant outgoing differential is

$$d_3 : E_3^{0,2m+2}(f)_v \to E_3^{3,2m}(f)_v.$$

Hence the proof is finished if we can show that $E_3^{3,2m}(f)_v = 0$. Clearly it suffices to show $E_2^{3,2m}(f)_v \cong H_c^3(U, \mathbb{V}_{2m}) = 0$.

The correspondence Γ'_* identifies the sheaf $(R^{2m} f_* \mathbb{Z})_v$ with $(\pi_* \mathbb{Z})_v$. As $H^4(S, \mathbb{Z}) \cong \mathbb{Z}$ and $H^3(S, \mathbb{Z}) = 0$ by hypothesis (i) and Poincaré duality, Lemma 3.1.8 shows that $E_2^{3,2m}(f)_v \cong H^3(W, \mathbb{Z})$. The statement then follows from Poincaré duality and Corollary 2.4.3, since hypothesis (i) implies that $H^1(S, \mathbb{Z}) = 0$. $\square$

The most obvious case where Theorem 3.1.13 applies is $S = \mathbb{P}^2$. In this case it was proved for a quadric bundle associated to a net of quadrics in odd–dimensional projective space by O'Grady [58], and in unpublished work of Beauville and Reid [loc. cit., p. 285]. Laszlo [43, Thm. II.3.1] proved Theorem 3.1.13 for arbitrary quadric bundles of even relative dimension over $\mathbb{P}^2$. Note that Laszlo and O'Grady state the result in a different form; they show that the transpose of Γ' induces an exact sequence

$$0 \to H^{2m+2}(\mathcal{X}, \mathbb{Z})_0 \xrightarrow{{}^t\Gamma'_*} H^2(W, \mathbb{Z})_0 \to \mathbb{Z}/2 \to 0. \tag{3.1}$$

If the sublattices $L_1 = \operatorname{im} \pi^* \subset H^2(W, \mathbb{Z})$ and $L_2 = \operatorname{im} i^* \subset H^{2m+2}(\mathcal{X}, \mathbb{Z})$ are primitive (this condition is satisfied in most cases, see [43, Prop. II.2.4.1] [58, Lemma 1.2]), the duality between variable and primitive cohomology (Lemma 2.2.5) shows that the version of Laszlo/O'Grady follows from our version by applying the functor $\mathrm{Hom}(\_, \mathbb{Z})$.

Remark 3.1.14 The case $m = 2$ of Theorem 3.1.13 is strongly related to a result of Mukai . Mukai shows that if X is a K3 surface and v a primitive isotropic Mukai vector on X, the moduli space $M = M(v)$ of stable sheaves with Mukai vector v is again a K3 surface. If X is a smooth complete intersection of three quadrics in $\mathbb{P}^5$ that corresponds to a regular quadric bundle, M is isomorphic to a double covering of $\mathbb{P}^2$ that is ramified along a smooth sextic; cf. [49, Example 2.2]

There exists an element $\alpha \in \mathrm{Br}(M)$ that measures the obstruction to the existence of a universal sheaf on $X \times M$ [14, §3.3]; if $\alpha = 0$ there exists a universal sheaf, and if α is n–torsion there exists a quasi–universal sheaf of similitude n (Mukai's terminology). This quasi–universal sheaf induces a cohomological Fourier–Mukai transformation $\varphi : T_X \to T_M$ on the transcendental lattices. This map is shown to be an embedding with cokernel $\mathbb{Z}/n$. In the example above one has $n = 2$; see [14, Example 5.1.13] and [42, §4.3–4.4]. If X and M are very general, the primitive and transcendental lattices coincide by the Noether–Lefschetz theorem, and the resulting exact sequence

$$0 \to T_X \xrightarrow{\varphi} T_M \to \mathbb{Z}/2 \to 0$$

gives a geometric interpretation of the cokernel $\mathbb{Z}/2$ appearing in the exact sequence (3.1).

Using results of Clemens and Steenbrink one can show that injectivity of Γ'_* fails for $r = 3$.

Proposition 3.1.15 *Let $f : \mathcal{X} \to \mathbb{P}^3$ be a regular quadric bundle, and let $\pi : W \to \mathbb{P}^3$ be the associated ramified double covering. Then*

$$\Gamma'_* : H^3(W, \mathbb{Q}) \to H^{2m+3}(\mathcal{X}, \mathbb{Q})$$

is not injective.

Proof: By Corollary 3.1.10 we have to verify that $H^3(W,\mathbb{Q})$ does not carry a pure Hodge structure of weight 3. The map $\pi : W \to \mathbb{P}^3$ is ramified over Δ_1, a surface of degree $2m+2$ with

$$k = \binom{2m+3}{3}$$

ordinary double points; cf. [39]. The threefold W has ordinary double points over the nodes of S. Put $\Sigma = \mathrm{Sing}(W)$. As the map π induces a bijection between the nodes of W and Δ_1, we shall identify Σ with $\Delta_2 = \mathrm{Sing}(\Delta_1)$. By [64, Thm. (1.13)] the group $H^4(W)$ carries a pure Hodge structure of weight four. As Σ consists of ordinary double points, [loc. cit., Cor. (1.12)] implies that $H^5_\Sigma(W) = 0$ and shows that the weight filtration $W_\bullet$ on $H^*_\Sigma(W)$ satisfies

$$\mathrm{Gr}^W_q\, H^3_\Sigma(W) = 0, \quad q \geq 3 \tag{3.2}$$
$$\mathrm{Gr}^W_q\, H^4_\Sigma(W) = 0, \quad q \leq 3. \tag{3.3}$$

Put $U = W \setminus \Sigma$ with inclusion map $j : U \to W$, and consider the exact sequence of local cohomology

$$H^2(U) \xrightarrow{\delta} H^3_\Sigma(W) \to H^3(W) \xrightarrow{j^*} H^3(U) \xrightarrow{\delta} H^4_\Sigma(W) \to H^4(W) \to H^4(U) \to 0.$$

As W is complete, $\mathrm{Gr}^W_q\, H^3(W) = 0$ for all $q \geq 4$. Hence $H^3(W)$ carries a pure HS of weight three if and only if $W_2H^3(W) = 0$. By (3.2) and strictness of the weight filtration, we have

$$W_2H^3(W) = \mathrm{im}(W_2H^3_\Sigma(W) \to H^3(W)), \quad W_2H^3_\Sigma(W) = \mathrm{im}\,\delta.$$

Hence

$$\begin{aligned} W_2H^3(W) = 0 \quad &\Longleftrightarrow \quad \delta : H^2(U) \to H^3_\Sigma(W) \;\text{ is surjective} \\ &\Longleftrightarrow \quad j^* : H^3(W) \to H^3(U) \;\text{ is injective.} \end{aligned}$$

Using Poincaré–Lefschetz duality on the smooth quasi–projective variety U we obtain

$$\dim_{\mathbb{C}} H^3(U) = \dim_{\mathbb{C}} H^3_c(U) = \dim_{\mathbb{C}} H^3(W,\Sigma) = \dim_{\mathbb{C}} H^3(W)$$

where the last equality follows since Σ is zero–dimensional. Hence

$$\begin{aligned} j^* \;\text{ is injective} \quad &\Longleftrightarrow \quad j^* \;\text{ is an isomorphism} \\ &\Longleftrightarrow \quad H^4_\Sigma(W) \cong \ker(j^* : H^4(W) \to H^4(U)). \end{aligned}$$

Using again Poincaré–Lefschetz duality, we obtain

$$\dim_{\mathbb{C}} \ker(j^* : H^4(W) \to H^4(U)) = b_4(W) - b_2(W) = b_4(W) - 1$$

where the last equality follows from the Barth–Lefschetz theorem for double coverings (Corollary 2.4.3).

By [18, p. 141] we have

$$b_4(W) - 1 = h^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}}(3m+3) \otimes \mathcal{I}_\Sigma).$$

This number measures to which extent the points of Σ fail to impose independent conditions on homogeneous polynomials of degree $3m+3$; it is called $\mathrm{def}(W)$, the defect of W.

A local computation [28, Chapter 6, Example (3.18)] shows that

$$\dim_{\mathbb{C}} H^4_\Sigma(W) = \mathrm{Card}(\Sigma).$$

Combining the previous results, we find that $H^3(W)$ carries a pure HS of weight 3 if and only if

$$\mathrm{def}(W) = \mathrm{Card}(\Sigma).$$

This means that the points of Σ impose no conditions on homogeneous polynomials of degree $3m+3$ on $\mathbb{P}^3$, which is absurd since $|\mathcal{O}_{\mathbb{P}}(3m+3)|$ is base–point free.

□

Remark 3.1.16 The result about the purity of the Hodge structure on $H^3(W)$ is known; see [55, Prop. 3.10] or [29, Cor. 2.8 (i)]. We have included a sketch of the proof since it nicely illustrates the difficulties that one encounters when W becomes singular.

3.2 Odd–dimensional quadrics

In this section we consider a quadric bundle $f : \mathcal{X} \to S$ of relative dimension $2m-1$. As in the previous situation, we consider the action of the correspondence defined by the universal family of m–planes contained in the fibers of f. To obtain such a family of m–planes we restrict to quadrics of corank at least one.

Put $\mathcal{X}_1 = \mathcal{X} \times_S \Delta_1$, and let

$$F_m(\mathcal{X}_1/\Delta_1) \xrightarrow{h} W_1 \xrightarrow{\pi_1} \Delta_1$$

be the Stein factorisation of the structure morphism $g : F_m(\mathcal{X}_1/\Delta_1) \to \Delta_1$. The map $\pi_1 : W_1 \to \Delta_1$ is a double covering with branch locus Δ_2. Let $p : \Gamma_1 \to F_m(\mathcal{X}_1/\Delta_1)$ be the universal family of m–planes, and let Γ_1' be its pullback to a subvariety $W_1' \subset W_1$ such that $h|_{W_1'}$ is finite. We have a commutative diagram

$$\begin{array}{ccccc} \Gamma_1' & \to & \mathcal{X}_1 & \to & \mathcal{X} \\ \downarrow & & \downarrow f_1 & & \downarrow f \\ W_1 & \xrightarrow{\pi_1} & \Delta_1 & \xrightarrow{i_1} & S. \end{array}$$

Put

$$\pi_1' = i_1 \circ \pi_1 : W_1 \to S.$$

By 3.1.3 the correspondence Γ_1' induces a homomorphism of constructible sheaves

$$\Gamma_{1,*}' : \pi_{1,*}'\mathbb{Z} \to R^{2m} f'\mathbb{Z}.$$

The following two results are the odd–dimensional analogues of Lemma 3.1.6 and Corollary 3.1.9. We omit the proofs.

Lemma 3.2.1 *The homomomorphism $\Gamma_{1,*}'$ induces an isomorphism*

$$(\pi_{1,*}'\mathbb{Z})_v \xrightarrow{\sim} (R^{2m} f_*\mathbb{Z})_v.$$

Corollary 3.2.2 *If $H^*(S,\mathbb{Z})$ is torison free and $H^r(\Delta_1,\mathbb{Z})$ has no 2–torsion there is an induced map*

$$\Gamma_{1,*}' : H^{r-1}(W_1,\mathbb{Z})_v \to H^{2m+r-1}(\mathcal{X},\mathbb{Z})_v$$

whose image is $E_\infty^{r-1,2m}(f)_v$. This map is injective if and only if $E_2^{r-1,2m}(f)_v = E_\infty^{r-1,2m}(f)_v$ and surjective if and only if $E_\infty^{r-2k-1,2m+2k}(f)_v = 0$ for all $k \geq 1$.

Theorem 3.2.3 *Let $f : \mathcal{X} \to S$ be a regular quadric bundle of relative dimension $2m-1$. The homomorphism*

$$\Gamma_{1,*}' : H^{r-1}(W_1,\mathbb{Q})_v \to H^{2m+r-1}(\mathcal{X},\mathbb{Q})_v$$

is an isomorphism if $r \leq 7$. If $r = 8$ then $\Gamma_{1,*}'$ is surjective.

Proof: As $\operatorname{codim} \Delta_3 = 6$ and $\operatorname{codim} \Delta_4 = 10$, Proposition 2.3.4 shows that only the local systems $\mathbb{V}_{2m} \to U_1$ and $\mathbb{V}_{2m+2} \to U_3$ contribute to the calculation of $H^{2m+r-1}(\mathcal{X},\mathbb{Z})_v$. Hence $\Gamma_{1,*}'$ is surjective if and only if $E_\infty^{r-3,2m+2}(f)_v = 0$. As $\dim \Delta_3 = r-6$ the corresponding E_2 term $H^3(\Delta_3, \mathbb{V}_{2m+2})$ vanishes if $r \leq 8$, since $r-3 > 2\dim \Delta_3$.

If $r \leq 9$, the homomorphism $\Gamma_{1,*}$ is injective if and only if the differential $d_3 : E_3^{r-4,2m+2}(f)_v \to E_3^{r-1,2m}(f)_v$ is zero. For $r \leq 8$ this condition is satisfied since $H^{r-4}(\Delta_3, \mathbb{V}_{2m+2}) = 0$, by an argument similar to the one above. $\square$

Remark 3.2.4 As before, one shows that the map $\Gamma'_{1,*}$ is injective if and only if $H^{r-1}(W_1)$ carries a pure Hodge structure of weight $r-1$. For $r=3$ there is a direct proof of this result. In this case W_1 is a double covering of a nodal surface Δ_1 that is ramified over the nodes. The surface W_1 is smooth; cf. [9, p. 131, footnote] and [17, Prop. 2.11].

Theorem 3.2.5 *Suppose that $H^*(S,\mathbb{Z})$ is torsion free. The homomorphism*

$$\Gamma'_{1,*} : H^{r-1}(W_1,\mathbb{Z}) \to H^{2m+r-1}(\mathcal{X},\mathbb{Z})_v$$

is an isomorphism if $r \leq 2$. If $r=3$ then $\Gamma'_{1,*}$ is injective.

Proof: If $r \leq 2$ then $E_\infty^{r-3,2m+2}(f)_v = 0$, hence $\Gamma_{1,*}$ is surjective over $\mathbb{Z}$. If $r \leq 3$ the differential the differential

$$d_3 : E_3^{r-4,2m+2}(f)_v \to E_3^{r-1,2m}(f)_v$$

vanishes, hence $\Gamma_{1,*}$ is injective over $\mathbb{Z}$. $\square$

Theorem 3.2.5 was proved by Beauville [8, Thm. 2.1] in the case $r=2$, $S=\mathbb{P}^2$. As an application of Theorem 3.2.5 we give a short proof of a theorem of Beltrametti and Francia [12, Thm. (3.5.2)], [13, Thm. 4.2.4 (i)]

Theorem 3.2.6 (Beltrametti–Francia) *Let $f : \mathcal{X} \to S$ be a regular conic bundle over a smooth surface S. Let $C \subset S$ be the discriminant curve, let $\tilde{C}$ be the associated étale double covering of C and let $\mathrm{Pr}(\tilde{C}/C)$ be the Prym variety of this covering. We have an exact sequence*

$$0 \to \mathrm{Pr}(\tilde{C}/C) \to J^2(\mathcal{X}) \to \mathrm{Alb}(S) \oplus \mathrm{Pic}^0(S) \to 0.$$

Proof: The conic bundle $\mathcal{X}$ is embedded in a projective bundle $\mathbb{P}(E)$, where E is a rank 3 vector bundle over S. As the spectral sequence

$$E_2^{p,q} = H^p(S,\mathbb{Z}) \otimes H^q(\mathbb{P}^2,\mathbb{Z}) \Rightarrow H^{p+q}(\mathbb{P}(E),\mathbb{Z})$$

degenerates at E_2, we obtain an isomorphism of Hodge structures

$$H^5(\mathbb{P}(E),\mathbb{Z}) \cong H^1(S,\mathbb{Z})(-2) \oplus H^3(S,\mathbb{Z})(-1).$$

Hence we obtain an exact sequence (after twisting by $\mathbb{Z}(1)$, cohomology with $\mathbb{Z}$–coefficients)

$$0 \to H^3(\mathcal{X})_0 \to H^3(\mathcal{X}) \xrightarrow{1_*} H^1(S)(-1) \oplus H^3(S) \to 0.$$

By Theorem 3.2.5 we have an isomorphism

$$\Gamma'_{1,*} : H^1(\tilde{C})_v \xrightarrow{\sim} H^3(\mathcal{X})_v,$$

and by duality we obtain an isomorphism

$$ {}^t\Gamma_{1,*} : H^3(\mathcal{X})_0 \to H^1(\tilde{C})_0. $$

Let $\iota : \tilde{C} \to \tilde{C}$ be the involution associated to the double covering $\pi : \tilde{C} \to C$. As $\pi^*\pi_* = \iota^* + \mathrm{id}$, we have $H^1(\tilde{C}, \mathbb{Z})_0 = H^1(\tilde{C}, \mathbb{Z})^-$. Applying the functor $\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}(-2), -)$ to the resulting exact sequence

$$ 0 \to H^1(\tilde{C})^-(-1) \to H^3(\mathcal{X}) \to H^1(S)(-1) \oplus H^3(S) \to 0 $$

we obtain a short exact sequence

$$ \begin{aligned} 0 \to \mathrm{Ext}^1_{MHS}(\mathbb{Z}(-1), H^1(\tilde{C})^-) &\to \mathrm{Ext}^1_{MHS}(\mathbb{Z}(-2), H^3(\mathcal{X})) \to \\ \to \mathrm{Ext}^1_{MHS}(\mathbb{Z}(-2), H^3(S)) &\oplus \mathrm{Ext}^1_{MHS}(\mathbb{Z}(-1), H^1(S)) \to 0 \end{aligned} $$

which gives the desired exact sequence using [15]. $\square$

Remark 3.2.7 Let $\mathcal{X} \to S$ be a quadric bundle of relative dimension n. If $n + \alpha$ is even, there exist a double covering $W_\alpha \to \Delta_\alpha$ that is ramified over $\Delta_{\alpha+1}$ and a correspondence Γ'_α that fit into a commutative diagram

$$ \begin{array}{ccccc} \Gamma'_\alpha & \to & \mathcal{X}_\alpha & \to & \mathcal{X} \\ \downarrow & & \downarrow{\scriptstyle f_\alpha} & & \downarrow{\scriptstyle f} \\ W_\alpha & \xrightarrow{\pi_\alpha} & \Delta_\alpha & \xrightarrow{i_\alpha} & S. \end{array} $$

The correspondence Γ'_α is given by a family of linear subspaces of dimension $\frac{n+\alpha}{2}$. Set $d_\alpha = \dim \Delta_\alpha$, and put

$$ p = \dim \mathcal{X} - \dim \Gamma'_\alpha = n + r - d_\alpha - \frac{n+\alpha}{2}. $$

Write $\pi'_\alpha = i_\alpha \circ \pi_\alpha$. By Lemma 3.1.3 the correspondence Γ'_α induces homomorphisms

$$ \pi'_{\alpha,*}\mathbb{Z} \to R^{2p} f_*\mathbb{Z}, \quad H^k(W_\alpha) \to H^{k+2p}(\mathcal{X}). $$

We have

$$ 2p = n + \alpha \Longleftrightarrow \mathrm{codim}\ \Delta_\alpha = \alpha. \tag{3.4} $$

If this equality holds, we obtain an isomorphism

$$ (\pi'_{\alpha,*}\mathbb{Z})_v \xrightarrow{\Gamma'_{\alpha,*}} \mathbb{V}_{n+\alpha}. $$

Hence the cohomology of the local system $\mathbb{V}_{n+\alpha}$ is related to the cohomology of the double covering W_α if and only if (3.4) holds. If this is the case, one also obtains a map

$$ E_2^{r-\alpha, n+\alpha}(f)_v \cong H^{r-\alpha}(W_\alpha)_v \xrightarrow{\Gamma'_{\alpha,*}} H^{n+r}(\mathcal{X})_v. $$

For a regular quadric bundle, (3.4) is satisfied if and only if $\alpha = 0$ or $\alpha = 1$. This explains the prominent rôle played by the correspondences Γ' (n even), Γ'_1 (n odd) and the local system $\mathbb{V}_{2m}$.

In Chapter 4 we shall consider quadric bundles equipped with an involution. For these quadric bundles Δ_2 contains an irreducible component $\Delta_{1,1}$ of codimension 2, which will play an important rôle in the calculation of the cohomology of the quadric bundle.

3.3 Generic surjectivity

As the dimension of the base S increases it becomes harder to analyse the behaviour of the spectral sequence $E_r^{p,q}(f)_v$, since more local systems come into play. Instead, one can prove surjectivity for general quadric bundles of a specific type using specialisation arguments.

Let G be the Grassmann variety of r–dimensional projective linear subspaces of $\mathbb{P}H^0(\mathbb{P}^{2m+1}, \mathcal{O}_{\mathbb{P}}(2))$. Given $t \in G$, we denote by $\mathbb{P}^r(t)$ the corresponding linear subspace. In the sequel we shall concentrate on the case $n = 2m$. There exists a commutative diagram

$$\begin{array}{ccc} \Gamma'_G & \rightarrow & \mathcal{X}_G \\ \downarrow & & \downarrow{\scriptstyle f_G} \\ W_G & \xrightarrow{\pi_G} & G \end{array}$$

whose fiber over $t \in G$ is

$$\begin{array}{ccc} \Gamma'(t) & \rightarrow & \mathcal{X}(t) \\ \downarrow & & \downarrow{\scriptstyle f(t)} \\ W(t) & \xrightarrow{\pi(t)} & \mathbb{P}^r(t). \end{array}$$

Put $P = \mathbb{P}H^0(\mathbb{P}^{2m+1}, \mathcal{O}_{\mathbb{P}}(2))$. The quadric bundle $\mathcal{X}(t)$ is the pullback of the universal quadric bundle $\mathcal{X}_{\text{univ}} \rightarrow P$ by the inclusion map $\mathbb{P}^r(t) \hookrightarrow P$. Let $\Delta_P \subset P$ be the discriminant locus of the universal quadric bundle with complement $U_P = P \setminus \Delta_P$. If $\mathcal{X}(t)$ is regular, its discriminant locus is

$$\Delta(t) = \Delta_P \cap \mathbb{P}^r(t)$$

with complement $U(t) = U_P \cap \mathbb{P}^r(t)$.

Lemma 3.3.1 *Suppose that there exists $t_0 \in G$ such that $\operatorname{im}\Gamma(t_0)_* \neq 0$. Then $\Gamma(t)_*$ is surjective for general $t \in G$.*

Proof: The map $\Gamma'(t)_*$ is $\pi_1(U(t))$–equivariant, since it is defined by a family of algebraic cycles over $\mathbb{P}^r(t)$. Hence $\operatorname{im}\Gamma'(t)_*$ is a $\pi_1(U(t))$–module. If $\mathcal{X}(t)$ is regular, the discriminant locus $\Delta(t) = \Delta_P \cap \mathbb{P}^r(t)$ is irreducible, hence the group $H^{2m+r}(\mathcal{X}(t), \mathbb{Q})_v$ is an irreducible $\pi_1(U(t))$–module. Hence it suffices to find $t \in G$ such that $\mathcal{X}(t)$ is regular and $\operatorname{im}\Gamma'(t)_* \neq 0$.

As the maps $f_G : \mathcal{X}_G \to G$ and $\pi_G : W_G \to G$ are proper, there exists a Zariski open subset $V \subset G$ such that $R^r\pi_{G,*}\mathbb{Q}$ and $R^{2m+r}f_{G,*}\mathbb{Q}$ are locally constant over V; see [68]. This implies that the correspondence

$$\Gamma'_{G,*} : R^r\pi_{G,*}\mathbb{Q}|_V \to R^{2m+r}f_{G,*}\mathbb{Q}|_V$$

is given by a matrix of locally constant functions. Hence

$$V' = \{t \in V | \mathcal{X}(t) \text{ is regular and } \operatorname{im}(\Gamma'(t)) \neq 0\}$$

is Zariski open in G. Put $\Sigma = G \setminus V'$. To show that V' is nonempty, we choose a line $\ell \subset G$ passing through t_0 such that $\ell \cap \Sigma$ is a finite set. There exists an open disc $D \subset \ell$ such that $t_0 \in D$ and $D \setminus \{t_0\} \subset V^0$. As $\mathcal{X}(t_0)$ is smooth, we may choose D such that $\mathcal{X}_D$ is smooth. There exist specialisation maps [36]

$$r(t) : W(t) \to W(t_0), \quad s(t) : \mathcal{X}(t) \to \mathcal{X}(t_0).$$

Consider the commutative diagram

$$\begin{array}{ccc} H^r(W(t_0)) & \xrightarrow{\Gamma'(t_0)_*} & H^{2m+r}(\mathcal{X}(t_0)) \\ \downarrow{\scriptstyle r(t)^*} & & \downarrow{\scriptstyle s(t)^*} \\ H^r(W(t)) & \xrightarrow{\Gamma'(t)_*} & H^r(W(t_0)). \end{array}$$

As $\mathcal{X}_D \to D$ is smooth, the map $s(t)^*$ is an isomorphism. Hence $\operatorname{im}\Gamma'(t_0)_* \neq 0$ implies $\operatorname{im}(\Gamma'(t))_* \neq 0$ and V' is nonempty. $\square$

It remains to find examples where the condition of Lemma 3.3.1 is satisfied. To this end, we consider a linear system of "diagonal" quadrics in $\mathbb{P}^{n+1}$ defined by

$$Q_i(x) = \sum_{j=0}^{n+1} a_{ij}x_j^2, \quad i = 0, \ldots, r.$$

Given $\lambda = (\lambda_0 : \ldots : \lambda_r) \in \mathbb{P}^r$, put $Q_\lambda = \sum_{i=0}^r \lambda_i Q_i$. This defines a quadric bundle $f : \mathcal{X} \to \mathbb{P}^r$ with $f^{-1}(\lambda) = Q_\lambda$. We have

$$Q_\lambda(x) = \sum_{j=0}^{n+1} \alpha_j x_j^2, \quad \alpha_j = \sum_i \lambda_i a_{ij}.$$

Let $H_j \subset \mathbb{P}^r$ be the hyperplane defined by the linear form α_j. If we choose a general matrix (a_{ij}), the arrangement of hyperplanes

$$\mathcal{A} = \{H_0, \ldots, H_{n+1}\}$$

is in general position and

$$D = \mathbb{P}^r \setminus \bigcup_{i=0}^{n+1} H_i$$

is a divisor with normal crossings. Given a multi–index $I \subset \{0, \ldots, n+1\}$, put

$$H_I = \cap_{i \in I} H_i, \quad \mathcal{A}_I = \{H_J \mid |J| = |I| + 1, J \supset I\} \subset H_I$$

Note that $\mathcal{A}_I \subset H_I$ is again a hyperplane arrangement in general position if the matrix (a_{ij}) is general. Write

$$M(\mathcal{A}) = \mathbb{P}^r \setminus D, \quad M(\mathcal{A}_I) = H_I \setminus \bigcup_{\substack{J \supset I \\ |J| = |I|+1}} H_J.$$

The stratification of $\mathcal{X} \to \mathbb{P}^r$ is given by

$$\Delta_i = \bigcup_{|I|=i} H_I.$$

Note that $U_i = \Delta_i \setminus \Delta_{i+1} = \coprod_{|I|=i} M(\mathcal{A}_I)$ is a disjoint union of complements of arrangements.

The fundamental group $\pi_1(M(\mathcal{A}_I))$ is abelian, hence

$$\pi_1(M(\mathcal{A}_I)) = H_1(M(\mathcal{A}_I)) \cong \mathbb{Z}^{n-|I|};$$

cf. [28, Chapter 4, Prop. (1.3) and Thm. (1.13)]. Choose a base point $0 \in M(\mathcal{A}_I)$, and let V be the fiber of the local system $\mathbb{V}_{n+i}$ at 0. We assume that $n+i$ is even. Let $G \subset \mathrm{Aut}(V)$ be the image of the monodormy representation

$$\rho : \pi_1(M(\mathcal{A}_I)) \to \mathrm{Aut}(V).$$

The monodromy group G is generated by operators T_j, $j \notin I$, that describe the monodromy around the hyperplane $H_{I \cup \{j\}} \subset H_I$, with one relation $\prod_{j \notin I} T_j = \mathrm{id}$.

Lemma 3.3.2 *If $n+i$ is even, then $T_j = -\mathrm{id}$ for all $j \notin I$.*

Proof: Without loss of generality we may assume that

$$I = \{n-i+2, \ldots, n+1\}, \quad j = 0.$$

Consider the pencil of quadrics defined by

$$Q_t(x) = tx_0^2 + x_1^2 + \ldots + x_{n-i+1}^2.$$

The quadric Q_t contains the linear subspaces

$$\begin{aligned}\Lambda'(t) &= V(\sqrt{t}x_0 + \sqrt{-1}x_1, x_2 + \sqrt{-1}x_3, \ldots, x_{n-i} + \sqrt{-1}x_{n-i+1})\\ \Lambda''(t) &= V(-\sqrt{t}x_0 + \sqrt{-1}x_1, x_2 + \sqrt{-1}x_3, \ldots, x_{n-i} + \sqrt{-1}x_{n-i+1}).\end{aligned}$$

Since the linear subspace

$$L = V(x_2 + \sqrt{-1}x_3, \ldots, x_{n-i} + \sqrt{-1}x_{n-i+1})$$

intersects Q_t in $\Lambda'(t) \cup \Lambda''(t)$, we have $[\Lambda'(t)] = -[\Lambda''(t)]$ in $H^{n+i}(Q_t, \mathbb{Z})$ $(t \neq 0)$. The monodromy operator T_0 acts by changing the sign of $\sqrt{t}$. Hence

$$T_0[\Lambda'(t)] = [\Lambda''(t)] = -[\Lambda'(t)].$$

□

For the proof of the main result of this section we need a calculation of the Euler characteristic of the complement of a hyperplane arrangement in general position.

Lemma 3.3.3 *Let $\mathcal{A} = \{H_i | i = 0, \ldots, \ell\}$ be a hyperplane arrangement in $\mathbb{P}^r$ with $\ell > r$. Set $M(\mathcal{A}) = \mathbb{P}^r \setminus \cup_i H_i$. If $\mathcal{A}$ is in general position, then*

$$e(M(\mathcal{A})) = (-1)^r \binom{\ell - 1}{r}.$$

Proof: If $\mathcal{A}$ is in general position, the divisor $D = H_0 \cup \ldots \cup H_\ell$ is a normal crossing divisor. Using Poincaré–Lefschetz duality one shows that $e(M(\mathcal{A})) = e_c(M(\mathcal{A}))$; hence $e(M(\mathcal{A})) = e(\mathbb{P}^r) - e(D)$. Using the Mayer–Vietoris spectral sequence

$$E_1^{p,q} = \oplus_{|I|=p} H^q(H_I, \mathbb{Q}) \Longrightarrow H^{p+q}(D, \mathbb{Q})$$

one shows that

$$e(D) = \sum_p (-1)^p \sum_{|I|=p} e(H_I).$$

Hence

$$e(M(\mathcal{A})) = r + 1 - r(\ell+1) + (r-1)\binom{\ell+1}{2} \ldots + (-1)^r \binom{\ell+1}{r}.$$

Using induction on r, one shows that this expression equals $(-1)^r\binom{\ell-1}{r}$.

□

Remark 3.3.4 In [20, Prop. 7.5] and [59, Lemma 5.122] this Euler characteristic calculation is carried out using a result of Hattori.

Theorem 3.3.5 *Let $f : \mathcal{X} \to \mathbb{P}^r$ be a quadric bundle associated to a general linear system of quadrics in $\mathbb{P}^{2m+1}$. The map*

$$\Gamma'_* : H^r(W, \mathbb{Q})_v \to H^{n+r}(\mathcal{X}, \mathbb{Q})_v$$

is surjective.

Proof: Let $f_0 : \mathcal{X}(t_0) \to \mathbb{P}^r(t_0)$ be a quadric bundle associated to a "diagonal" linear system of quadrics defined by a general matrix (a_{ij}). By Lemma 3.3.2 and Proposition 2.4.5 we have

$$H^q_c(U_i, \mathbb{V}_{2m+i}) = 0 \quad \text{if } q \neq \dim U_i = r - i.$$

Hence $E_2^{r-i-1,2m+i}(f_0))_v = 0$ and the differentials

$$d_{i+1} : E_{i+1}^{r-i-1,2m+i}(f_0)_v \to E_{i+1}^{r,2m}(f_0)_v$$

vanish for all $i \geq 1$. This implies that

$$E_\infty^{r,2m}(f_0)_v = E_2^{r,2m}(f_0)_v = H^r_c(U, \mathbb{V}_{2m}).$$

As $H^q_c(U, \mathbb{V}_{2m}) = 0$ for all $q \neq r$ by Proposition 2.4.5, we have

$$\text{rank } H^r_c(U, \mathbb{V}_{2m}) = |e_c(U). \text{rank}\,(\mathbb{V}_{2m})| = |e_c(U)|.$$

By Lemma 3.3.3 we have $|e_c(U)| = |e(U)| = \binom{2m}{r} \neq 0$. Hence $E_\infty^{r,2m}(f_0) \neq 0$ and $\Gamma(t_0)_*$ is surjective by Corollary 3.1.9. $\square$

The previous example admits a natural generalisation. Consider the vector bundle $E = \mathcal{O}_\mathbb{P}(-d_0) \oplus \ldots \oplus \mathcal{O}_\mathbb{P}(-d_{n+1})$ over $\mathbb{P}^r$, and put $L = \mathcal{O}_\mathbb{P}(d)$. The variety $P = \mathbb{P}(E)$ is a smooth toric variety. According to results of D. Cox [23] it has a homogeneous coordinate ring

$$R = \mathbb{C}[x_0, \ldots, x_r, y_0, \ldots, y_{n+1}]$$

that admits a grading by $\text{Pic}(P) \cong \mathbb{Z}^2$. The variables x_i have bidegree $(0, 1)$ and y_j has bidegree $(1, -d_j)$ [30] Consider the quadratic form

$$q = \sum_i f_i(x) y_i^2$$

with $\deg(f_i) = d + 2d_i$. Let $\psi : P \to \mathbb{P}^r$ be the projection map. As q is a form of bidegree $(2, d)$, it corresponds to a section of $\xi_E^2 \otimes \psi^* L$. By choosing general polynomials f_i, we may assume that the discriminant locus $D = \cup_i V(f_i)$ is a simple normal crossing divisor. Using a Chern polynomial calculation one can show that the Euler characteristic of $\mathbb{P}^r \setminus D$ is nonzero if the degrees d_i are sufficiently large. Using Proposition 2.4.5 the previous argument goes through in this situation. The previous example corresponds to the case $d_0 = \ldots = d_{n+1} = 1$, $d = -1$.

Recall that the generalised Hodge conjecture $\mathrm{GHC}(X, m, p)$ states that

$$N^p H^m(X, \mathbb{Q}) = \bigcup_{\substack{Z \subset X \\ \mathrm{codim}\ Z \geq p}} \mathrm{im}(H^m_Z(X, \mathbb{Q}) \to H^m(X, \mathbb{Q}))$$

is the largest $\mathbb{Q}$–sub Hodge structure contained in $F^p H^m(X, \mathbb{C}) \cap H^m(X, \mathbb{Q})$; cf. [65] and [45].

Proposition 3.3.6 *Let $f : \mathcal{X} \to S$ be a quadric bundle of relative dimension $2m$. Then* $\mathrm{GHC}(X, 2m + r, m)$ *holds in the following cases.*

(i) $\dim S \leq 3$ *and $\mathcal{X}$ is a regular quadric bundle;*

(ii) $\mathcal{X} \to \mathbb{P}^r$ is a quadric bundle associated to a general linear system of quadrics.

Proof: If $\Gamma \in \mathrm{Corr}^k(X, Y)$ induces a surjective map $\Gamma_* : H^n(X) \to H^{n+2k}(Y)$ then $\mathrm{GHC}(X, n, p)$ implies $\mathrm{GHC}(Y, n + 2k, p + k)$; cf. [4]. Hence the statement follows from $\mathrm{GHC}(W, r, 0)$ (which is trivially satisfied) using Theorems 3.1.11 and 3.3.5. □

The proof of the following result is analogous to the proof of Theorem 3.3.5.

Theorem 3.3.7 *Let $\mathcal{X} \to \mathbb{P}^r$ be a quadric bundle of relative dimension $2m-1$ associated to a general linear system of quadrics. Then*

$$\Gamma'_{1,*} : H^{r-1}(W) \to H^{2m+r-1}(\mathcal{X}, \mathbb{Q})$$

is surjective.

Corollary 3.3.8 *Let $\mathcal{X} \to S$ be a quadric bundle of relative dimension $2m-1$. Then* $\mathrm{GHC}(\mathcal{X}, 2m + r - 1, m)$ *holds in the following cases.*

(i) $\mathcal{X}$ is regular and $\dim S \leq 8$*;*

(ii) $\mathcal{X} \to \mathbb{P}^r$ is associated to a general linear system of quadrics.

Proof: Use Theorems 3.2.3 and 3.3.7. □

Remark 3.3.9 Using the Cayley trick, case (ii) of Proposition 3.3.6 and Corollary 3.3.8 gives a new proof of a result of Shimada [63] on the GHC for complete intersections of quadrics, which was obtained using a mixture of topological and infinitesimal techniques.

Chapter 4

Quadric bundles with involution and applications

The motivation for this chapter was the following example of Bardelli [6]. Let $\sigma : \mathbb{P}^7 \to \mathbb{P}^7$ be the involution defined by $\sigma(x_0 : \ldots : x_3 : y_0 : \ldots : y_3) = (x_0 : \ldots : x_3 : -y_0 : \ldots : y_3)$, and let $X = V(Q_0, \ldots, Q_3)$ be a smooth complete intersection of four σ–invariant quadrics. Bardelli shows that there exists a smooth curve C of genus 33, obtained as the complete intersection of two nodal surfaces S', S'' in $\mathbb{P}^3$, and an étale double covering $\tilde{C} \to C$. such that

$$H^1(\tilde{C}, \mathbb{Q})^- \cong H^3(X, \mathbb{Q})^-.$$

We shall see that there is a natural interpretation in terms of quadric bundles with involution, which yields a simple proof of this result.

4.1 Quadric bundles with involution

Let S be a smooth projective variety of dimension r, and let E be a holomorphic rank $2m+2$ vector bundle over S with projection map $p : E \to S$. Suppose that E admits an involution τ such that $p \circ \tau = p$. The action of τ gives a decomposition $E = E^+ \oplus E^-$. For the induced action on $S^2 E^\vee$ we have

$$(S^2 E^\vee)^+ = (S^2 E^+)^\vee \oplus (S^2 E^-)^\vee.$$

Let L be a line bundle on S on which τ acts trivially, and let $f : \mathcal{X} \to S$ be the quadric bundle associated to a τ–invariant quadratic form $q \in H^0(S, S^2 E^\vee \otimes L)$. As q is τ–invariant, it corresponds to a pair

$$(q', q'') \in H^0(S, (S^2 E^+)^\vee \otimes L) \oplus H^0(S, (S^2 E^-)^\vee \otimes L).$$

Define

$$\Delta_k \;=\; \{\lambda \in S \mid \text{corank } q(\lambda) \geq k\}$$

$$
\begin{aligned}
\Delta_{i,j} &= \{\lambda \in S \mid \operatorname{corank} q'(\lambda) \geq i, \quad \operatorname{corank} q''(\lambda) \geq j\} \\
U_k &= \Delta_k \setminus \Delta_{k+1} \\
U_{i,j} &= \Delta_{i,j} \setminus (\Delta_{i+j+1} \cap \Delta_{i,j}) = \Delta_{i,j} \setminus (\Delta_{i+1,j} \cup \Delta_{i,j+1}).
\end{aligned}
$$

Note that we have a decompositions

$$
\Delta_k = \bigcup_{i+j=k} \Delta_{i,j}, \quad U_k = \bigcup_{i+j=k} U_{i,j}.
$$

Definition 4.1.1 *A τ–invariant quadric bundle $\mathcal{X} \to S$ is called* τ–regular *if*

(i) $\Delta_{i,j}$ *is irreducible of codimension* $\binom{i+1}{2} + \binom{j+1}{2}$ *if* $\Delta_{i,j}$ *is nonempty;*

(ii) $\operatorname{Sing} \Delta_{i,j} = \Delta_{i+1,j} \cup \Delta_{i,j+1}$.

From now on we assume that the action of τ on $\mathbb{P}(E)$ is nontrivial, i.e., $E^+ \neq 0$ and $E^- \neq 0$. Write $\mu = \operatorname{rank}(E^+) - 1$, $\nu = \operatorname{rank}(E^-) - 1$. We have $\mu + \nu = 2m$, hence μ is even if and only if ν is even.

Proposition 4.1.2 *Let $[\Lambda'], [\Lambda''] \in H^{2m+2k}(Q_\lambda, \mathbb{Z})$ be the generators of the cohomology of a quadric Q_λ of even corank $2k$.*

(i) If $\lambda \in \Delta_{i,j} \subset \Delta_{2k}$ with i even, the action of τ^* on $H^{2m+2k}(Q_\lambda, \mathbb{Z})$ is given by

$$
\begin{aligned}
\tau^*[\Lambda'] &= [\Lambda'], \quad \tau^*[\Lambda''] = [\Lambda''] \text{ if } \mu \text{ is odd;} \\
\tau^*[\Lambda'] &= [\Lambda''], \quad \tau^*[\Lambda''] = [\Lambda'] \text{ if } \mu \text{ is even.}
\end{aligned}
$$

(ii) If $\lambda \in \Delta_{i,j} \subset \Delta_{2k}$ with i odd, the action if τ^* on $H^{2m+2k}(Q_\lambda, \mathbb{Z})$ is given by

$$
\begin{aligned}
\tau^*[\Lambda'] &= [\Lambda''], \quad \tau^*[\Lambda''] = [\Lambda'] \text{ if } \mu \text{ is odd;} \\
\tau^*[\Lambda'] &= [\Lambda'], \quad \tau^*[\Lambda''] = [\Lambda''] \text{ if } \mu \text{ is even.}
\end{aligned}
$$

Proof: There exist local coordinates on $\mathbb{P}^{2m+1}_\lambda$ such that the action of τ is given by

$$
\tau(x_0, \ldots, x_\mu, y_0, \ldots, y_\nu) = (x_0, \ldots, x_\mu, -y_0, \ldots, -y_\nu).
$$

We may assume that Q'_λ and Q''_λ are in diagonal form and

$$
Q_\lambda = V(x_i^2 + \ldots + x_\mu^2 + y_j^2 + \ldots + y_\nu^2).
$$

We start with (i). If μ is odd, we take

$$\begin{aligned}\Lambda' &= V(x_i + \sqrt{-1}x_{i+1}, \ldots, x_{\mu-1} + \sqrt{-1}x_\mu, y_j + \sqrt{-1}y_{j+1}, \ldots, y_{\nu-1} + \sqrt{-1}y_\nu)\\ \Lambda'' &= V(x_i - \sqrt{-1}x_{i+1}, \ldots, x_{\mu-1} + \sqrt{-1}x_\mu, y_j + \sqrt{-1}y_{j+1}, \ldots, y_{\nu-1} + \sqrt{-1}y_\nu).\end{aligned}$$

These two linear subspaces intersect in codimension one, hence they belong to different rulings. As the equations of Λ' and Λ'' are τ–invariant, their cycle classes are fixed by τ^*. If μ is odd, we take

$$\begin{aligned}\Lambda' &= V(x_i + \sqrt{-1}y_j, x_{i+1} + \sqrt{-1}x_{i+2}, \ldots, x_{\mu-1} + \sqrt{-1}x_\mu,\\ &\quad y_{j+1} + \sqrt{-1}y_{j+2}, \ldots, y_{\nu-1} + \sqrt{-1}y_\nu)\\ \Lambda'' &= V(x_i - \sqrt{-1}y_j, x_{i+1} + \sqrt{-1}x_{i+2}, \ldots, x_{\mu-1} + \sqrt{-1}x_\mu,\\ &\quad y_{j+1} + \sqrt{-1}y_{j+2}, \ldots, y_{\nu-1} + \sqrt{-1}y_\nu).\end{aligned}$$

In this case τ interchanges the equations of Λ' and Λ'', hence τ^* interchanges their cycle classes.

The proof of (ii) proceeds in a similar way and will be omitted. $\square$

Table 4.1.3 (μ odd)

$\mathbb{V}$	$\mathrm{Supp}(\mathbb{V})$	$\mathrm{codim}\,(\mathrm{Supp}\,\mathbb{V})$
$\mathbb{V}^+_{2m}$	S	*0*
$\mathbb{V}^+_{2m+2}$	$\Delta_{0,2} \cup \Delta_{2,0}$	*3*
$\mathbb{V}^+_{2m+4}$	$\Delta_{0,4} \cup \Delta_{2,2} \cup \Delta_{4,0}$	*6*
$\mathbb{V}^-_{2m+2}$	$\Delta_{1,1}$	*2*
$\mathbb{V}^-_{2m+4}$	$\Delta_{1,3} \cup \Delta_{3,1}$	*7.*

The corresponding table for μ even is obtained by interchanging $\mathbb{V}^+_q$ by $\mathbb{V}^-_q$.

For the remainder of this chapter we consider a τ–regular quadric bundle with involution $f : \mathcal{X} \to S$ of relative dimension $2m$. As before, the Stein factorisation of the map $F_m(\mathcal{X}/S)$ defines a double covering

$$\pi : W \to S$$

that is ramified over $\Delta_1 = \Delta_{1,0} \cup \Delta_{0,1}$. Put $\mathcal{X}_{1,1} = \mathcal{X} \times_S \Delta_{1,1}$. The Stein factorisation of the map $F_{m+1}(\mathcal{X}_{1,1}/\Delta_{1,1}) \to \Delta_{1,1}$ defines a double covering

$$\pi_{1,1} : W_{1,1} \to \Delta_{1,1}$$

that is ramified over $\operatorname{Sing}\Delta_{1,1} = \Delta_{2,1} \cup \Delta_{1,2}$. There exist subvarieties $W' \subset F_m(\mathcal{X}/S)$ resp.$W'_{1,1} \subset F_{m+1}(\mathcal{X}_{1,1}/\Delta_{1,1})$ that are étale over W resp. $W_{1,1}$. Let Γ', $\Gamma'_{1,1}$ be the pullbacks of the universal families

$$\Gamma \to F_m(\mathcal{X}/S), \quad \Gamma_{1,1} \to F_{m+1}(\mathcal{X}_{1,1}/\Delta_{1,1})$$

to these subvarieties. The situation can be summarised by the commutative diagram

$$\begin{array}{ccccccc}
\Gamma'_{1,1} & \to & \mathcal{X}_{1,1} & \to & \mathcal{X} & \leftarrow & \Gamma' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
W_{1,1} & \xrightarrow{\pi_{1,1}} & \Delta_{1,1} & \xrightarrow{i_{1,1}} & S & \xleftarrow{\pi} & W.
\end{array}$$

Put

$$\pi'_{1,1} = i_{1,1} \circ \pi_{1,1}.$$

By Corollary 3.1.4 we obtain homomorphisms

$$\Gamma'_* : \pi_*\mathbb{Z} \to R^{2m} f_*\mathbb{Z}, \quad (\Gamma'_{1,1})_* : ((\pi'_{1,1})_*\mathbb{Z} \to R^{2m+2} f_*\mathbb{Z}.$$

Lemma 4.1.4 *There exist spectral sequences $E_r^{p,q}(f)_v^{\pm} \Rightarrow H^{p+q}(\mathcal{X}, \mathbb{Q})_v^{\pm}$ with E_2 terms*

$$E_2^{p,q}(f)_v^+ = H^p(S, (R^q f_*\mathbb{Q})_v^+), \quad E_2^{p,q}(f)_v^- = H^p(S, (R^q f_*\mathbb{Q})_v^-).$$

Proof: The differentials in the spectral sequence $E_r^{p,q}(f)_v$ are equivariant with respect to the action of τ^*. Hence the decomposition $R^q f_*\mathbb{Q} \cong (R^q f_*\mathbb{Q})^+ \oplus (R^q f_*\mathbb{Q})^-$ induces isomorphisms $E_r^{p,q}(f)_v \cong E_r^{p,q}(f)_v^+ \oplus E_r^{p,q}(f)_v^-$ and the differentials

$$d_r : E_r^{p,q}(f)_v^+ \to E_r^{p+r,q-r+1}(f)_v^-, \quad d_r : E_r^{p,q}(f)_v^- \to E_r^{p+r,q-r+1}(f)_v^+$$

are zero. $\square$

Proposition 4.1.5 *(i) If μ is odd, there exist maps*

$$\begin{aligned}
\Gamma'_* &: H^r(W, \mathbb{Q})^- \to H^{2m+r}(\mathcal{X}, \mathbb{Q})_v^+ \\
(\Gamma'_{1,1})_* &: H^{r-2}(W_{1,1}, \mathbb{Q})^- \to H^{2m+r}(\mathcal{X}, \mathbb{Q})_v^-.
\end{aligned}$$

(ii) If μ is even, there exist maps

$$\begin{aligned}
\Gamma'_* &: H^r(W, \mathbb{Q})^- \to H^{2m+r}(\mathcal{X}, \mathbb{Q})_v^- \\
(\Gamma'_{1,1})_* &: H^{r-2}(W_{1,1}, \mathbb{Q})^- \to H^{2m+r}(\mathcal{X}, \mathbb{Q})_v^+.
\end{aligned}$$

Proof: We give the proof of (i); the proof of (ii) is similar. By Proposition 4.1.2 and the proper base change theorem, we find that Γ'_* induces an isomorphism

$$\Gamma'_* : (\pi_*\mathbb{Z})_v \to (R^{2m} f_*\mathbb{Z})^+_v$$

The resulting map $H^r(S, (\pi_*\mathbb{Q})_v) \to H^r(S, (R^{2m} f_*\mathbb{Q})^+_v)$ gives a homomorphism

$$\Gamma'_* : H^r(W, \mathbb{Q})^- \to H^{2m+r}(\mathcal{X}, \mathbb{Q})^+_v$$

that factors through $E^{r,2m}_\infty(f)^+_v$, as in the proof of Corollary 3.1.9.

A similar argument shows that $\Gamma'_{1,1}$ induces an isomorphisms

$$((\pi_{1,1})'_*\mathbb{Z})_v \xrightarrow{\sim} (R^{2m+2}(f_{1,1})_*\mathbb{Z})_v$$
$$H^{r-2}(W_{1,1}, \mathbb{Q})^- \xrightarrow{\sim} E^{r-2,2m+2}_2(f)^-_v.$$

As $E^{p,q}_r(f)^-_v = 0$ for all $q < 2m+2$ by Proposition 4.1.2 there are no outgoing differentials at position $(r-2, 2m+2)$, hence the map $E^{r-2,2m+2}_2(f)^-_v \to E^{r-2,2m+2}_\infty(f)^-_v$ is surjective. Composition of the previous maps with the inclusion $E^{r-2,2m+2}_\infty(f)^-_v \subset H^{2m+r}(\mathcal{X}, \mathbb{Q})^-_v$ gives a homomorphism

$$(\Gamma'_{1,1})_* : H^{r-2}(W_{1,1}, \mathbb{Q})^- \to H^{2m+r}(\mathcal{X}, \mathbb{Q})^-_v.$$

$\square$

Theorem 4.1.6 *Let $\mathcal{X} \to S$ be a τ–regular quadric bundle with involution of relative dimension $2m$. The map*

$$\Gamma'_* : H^r(W, \mathbb{Q})^- \to H^{2m+r}(\mathcal{X}, \mathbb{Q})^\pm_v$$

is an isomorphism if $r \leq 2$. If $r = 3$ then Γ'_* is surjective.

Proof: As usual we give the proof in case μ is odd, the other case being similar. If $r \leq 2$ then $E^{p,q}_r(f)^+_v = 0$ for all $q \neq 2m$, since the local systems $\mathbb{V}_{2m+2k}$ are supported on (subsets of) $\Delta_{2,0} \cup \Delta_{0,2}$ and both subvarieties have codimension 3. If $r \leq 5$ the only local systems that contribute to $H^{2m+r}(\mathcal{X}, \mathbb{Q})^+$ are $\mathbb{V}^+_{2m}$ and $\mathbb{V}^+_{2m+2}$. Hence it suffices to show that $E^{r-2,2m+2}_\infty(f)^+_v = 0$ By Proposition 4.1.2 we have

$$E^{r-2,2m+2}_2(f)^+_v = H^{r-2}_c(U_{2,0}, \mathbb{V}^+_{2m+2}) \oplus H^{r-2}_c(U_{0,2}, \mathbb{V}^+_{2m+2}).$$

As $\dim U_{0,2} = \dim U_{2,0} = r-3$ the result is clear if $r \leq 2$. If $r = 3$ then $\Delta_{2,0}$ and $\Delta_{0,2}$ are finite sets, hence $H^1(\Delta_{i,j}, \mathbb{V}^+_{2m+2}) = 0$, $(i,j) \in \{(0,2), (2,0)\}$, and $E^{1,2m+2}_2(f)^+_v = 0$. $\square$

Theorem 4.1.7 *The map*

$$(\Gamma'_{1,1})_* : H^{r-2}(W_{1,1}, \mathbb{Q})^- \to H^{2m+r}(\mathcal{X}, \mathbb{Q})^{\pm}_v$$

is an isomorphism if $r \leq 8$. If $r = 9$ then $(\Gamma'_{1,1})_*$ is surjective.

Proof: Again we only treat the case μ odd. As $\mathbb{V}^-_{2m+4}$ is supported on a subvariety of codimension 7, we have $E_2^{r-4,2m+4}(f)^- = H^{r-4}(S, \mathbb{V}^-_{2m+4}) = 0$ if $r - 4 > 2(r - 7)$, i.e., $r < 10$.

For the injectivity we note that $E_2^{r-5,2m+4}(f)^- = 0$ if $r \leq 8$ by the same dimension argument. □

Bardelli's theorem is an immediate consequence of the case $r = 3$, $\mu = 3$, $m = 3$ of Theorem 4.1.7. Indeed, let $\mathcal{X} \to \mathbb{P}^3$ be the τ–invariant quadric bundle associated to the τ–invariant Calabi–Yau threefold $X = V(Q_0, \ldots, Q_3)$. The stratification of the base gives us two surfaces $\Delta_{1,0}$, $\Delta_{0,1}$, each with then ordinary double points (Card($\Delta_{2,0}$ = Card($\Delta_{0,2}$) = 10) that intersect in a smooth curve $\Delta_{1,1}$ of genus 33. Combining the isomorphism of Theorem 4.1.7 with the Cayley trick, we obtain Bardelli's theorem. Bardelli's original proof was obtained using a combination of infinitesimal computations and monodromy techniques.

Remark 4.1.8 Using an ingenious degeneration argument, Bardelli proves a stronger result: the image of the map

$$H^1(W_{1,1}, \mathbb{Z})^- \to H^3(X, \mathbb{Z})^-$$

is a subgroup of index 2. This result is related to the behaviour of the differential

$$d_3 : E_3^{1,8}(f)^- \to E_3^{4,6}(f)^+.$$

As the map d_3 is τ–equivariant, it is clear $T = \mathrm{im}(d_3)$ is a 2–torsion group. Bardelli's result says that $T \cong \mathbb{Z}/2$.

A straightforward generalisation of the degeneration argument of Chapter 3 gives the following analogue of Theorem 3.3.5.

Theorem 4.1.9 *Let $\tau : \mathbb{P}^{2m+1} \to \mathbb{P}^{2m+1}$ be a nontrivial involution defined by*

$$\tau(x_0 : \ldots : x_\mu : y_0 : \ldots : y_\nu) = (x_0 : \ldots : x_\mu : -y_0 : \ldots : -y_\nu),$$

and let $\mathcal{X} \to \mathbb{P}^r$ be the quadric bundle with involution of associated to a general $r + 1$–dimensional linear subspace of $H^0(\mathbb{P}^{2m+1}, \mathcal{O}_{\mathbb{P}}(2))^+$.

(i) If μ is odd, the map

$$(\Gamma'_{1,1})_* : H^{r-2}(W_{1,1}, \mathbb{Q})^- \to H^{2m+r}(\mathcal{X}, \mathbb{Q})_v^-$$

is surjective;

(ii) If μ is even, the map

$$\Gamma'_* : H^r(W, \mathbb{Q})^- \to H^{2m+r}(\mathcal{X}, \mathbb{Q})_v^-$$

is surjective.

Proof: Argue as in Theorem 3.3.5, using the irreducibility of the monodromy action on $H^{2m+r}(\mathcal{X}, \mathbb{Q})_v^-$. The latter result follows from Picard–Lefschetz theory; cf. [70, Lemme 2.19]. □

Remark 4.1.10 As it is not known whether the monodromy action on $H^{2m+r}(\mathcal{X}, \mathbb{Q})_v^+$ is irreducible, it is not clear whether the maps with target $H^{2m+r}(\mathcal{X}, \mathbb{Q})_v^+$ are generically surjective. The problem is that the discriminant locus in $\mathbb{P}H^0(\mathbb{P}^{2m+1}, \mathcal{O}_{\mathbb{P}}(2))^+$ has two irreducible components that both contribute to the vanishing cycles in $H^{2m+r}(\mathcal{X}, \mathbb{Q})_v^+$. In [6], Bardelli shows the irreducibility of the monodromy action for his example ($m = 3$, $\mu = 3$). It is not clear whether his argument can be extended to the general case.

4.2 Relationship with Nori's theorem

It is known that the vanishing of the relative cohomology $H^*(\mathbb{P}^N_T, X_T)$ puts strong restrictions on the image of the regulator maps

$$c_{p,q} : \mathrm{CH}^p(X, q) \to H^{2p-q}_{\mathcal{D}}(X, \mathbb{Z}(p)).$$

Specifically, the image of these maps is contained in the image of the restriction map $i^* : H^{2p-q}_{\mathcal{D}}(\mathbb{P}^N, \mathbb{Z}(p)) \to H^{2p-q}_{\mathcal{D}}(X, \mathbb{Z}(p))$ if X is a very general complete intersection of sufficiently high multidegree. The case $q = 0$ corresponds to the theorems of Noether–Lefschetz and Green–Voisin on the image of the cycle class and Abel–Jacobi maps; cf. [52].

The effective degree bounds computed in [51] lead to a number of examples of low multi–degree where the statement of Nori's connectivity theorem is not known to hold. These examples typically involve intersections of quadrics. After closer examination, three cases remain:

(i) $X = V(2, 2, 2, 2) \subset \mathbb{P}^{2m+1}$, $q = 0$;

(ii) $X = V(2,2,2) \subset \mathbb{P}^{2m+1}$, $q = 1$;

(iii) $X = V(2,2) \subset \mathbb{P}^{2m+1}$, $q = 2$.

All three cases can be settled using the techniques for dealing with the cohomology of quadric bundles presented in the previous chapters. We shall see that the conclusion of Nori's theorem fails in the first case, whereas it holds in the two other cases.

4.3 Abel–Jacobi maps

In this section we study the first case, complete intersections of four quadrics in an odd–dimensional projective space.

Theorem 4.3.1 *Let X be a smooth complete intersection of four quadrics in $\mathbb{P}^{2m+1}$. If X is general, then the image of the Abel–Jacobi map*

$$\mathrm{AJ}_X : \mathrm{CH}^{m-1}_{\mathrm{hom}}(X) \to J^{m-1}(X)$$

is nonzero. If X is very general, the image of AJ_X is not contained in the torsion points of $J^{m-1}(X)$.

Proof: Write $\mathbb{P}^{2m+1} = \mathbb{P}(V)$. A complete intersection of four quadrics is the base locus of a web of quadrics $M \cong \mathbb{P}^3$. Let $T \subset G(4, S^2V^\vee)$ be the open subset that parametrises webs of quadrics whose base locus is smooth. Choose coordinates $(\underline{x}, \underline{y}) = (x_0 : \ldots : x_m : y_0 : \ldots : y_m)$ on $\mathbb{P}^{2m+1}$ and consider the involution

$$\sigma : \mathbb{P}^{2m+3} \to \mathbb{P}^{2m+3}, \sigma(\underline{x}, \underline{y}) = (\underline{x}, -\underline{y}).$$

Given $t \in G(4, S^2V^\vee)$, we write $M(t)$ for the corresponding web of quadrics, $X(t) = \mathrm{Bs}(M(t))$ for the corresponding complete intersection and $f(t) : \mathcal{X}(t) \to M(t)$ for the associated quadric bundle of relative dimension $2m$. Over the locus $\Delta_2(t)$ of corank 2 quadrics we have a family of $(m+1)$–planes. As before, this defines a double covering $W_2(t) \to \Delta_2(t)$ and a correspondence Γ_2 of degree $m+2$ that induces a homomorphism

$$\Gamma_{2,*}(t) : \mathrm{CH}_0(W_2(t)) \to \mathrm{CH}^{m+2}(\mathcal{X}(t)).$$

Note that there is a natural inclusion $i(t) : X(t) \times M(t) \hookrightarrow \mathcal{X}(t)$. Hence we obtain a homomorphism

$$\varphi(t) : \mathrm{CH}^{m+2}(\mathcal{X}(t)) \xrightarrow{i(t)^*} \mathrm{CH}^{m+2}(X(t) \times M(t)) \xrightarrow{p_{1,*}(t)} \mathrm{CH}^{m-1}(X(t)).$$

Let $\lambda'(t)$, $\lambda''(t)$ be the two points in the fiber over $\lambda(t) \in W_2(t)$. The composition $\varphi(t) \circ \Gamma_{2,*}(t)$ maps $[\lambda'(t) - \lambda''(t)]$ to the cycle

$$Z(t) = [\Lambda'(t) \cap X(t) - \Lambda''(t) \cap X(t)]$$

which is homologically equivalent to zero. Hence we obtain a homomorphism

$$\psi(t) : \mathrm{CH}_0(W_2(t))^- \to \mathrm{CH}^{m-1}_{\mathrm{hom}}(X(t)).$$

Choose $t_0 \in G(4, S^2V^\vee)$ such that $M(t_0)$ is a σ–regular web of σ–invariant quadrics. In this case the web $M(t_0)$ contains a smooth curve $\Delta_{1,1}(t_0)$, and we obtain as before a double covering $W_{1,1}(t_0) \to \Delta_{1,1}(t_0)$ and a correspondence $\Gamma_{1,1}(t_0)$ of degree $m+1$. We obtain a commutative diagram

$$\begin{array}{ccccc}
\mathrm{CH}_0(W_{1,1}(t_0))^- & \xrightarrow{(\Gamma_{1,1}(t_0))_*} & \mathrm{CH}^{m+2}_{\mathrm{hom}}(\mathcal{X}(t_0))^\pm & \to & \mathrm{CH}^{m-1}_{\mathrm{hom}}(X(t_0))^\pm \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Pr}(W_{1,1}(t_0)/\Delta_{1,1}(t_0)) & \to & J^{m+2}(\mathcal{X}(t_0))^\pm & = & J^{m-1}(X(t_0))^\pm .
\end{array}$$

Note that the total space of the family

$$I = \{(Q, t) \mid Q \in \Delta_2 \cap M(t)\}$$

is irreducible, since Δ_2 and the fibers of $p_1 : I \to \mathbb{P}(S^2V^\vee)$ are irreducible. Hence we can connect $\lambda(t) \in \Delta_2(t)$ and $\lambda(t_0) \in \Delta_{1,1}(t_0)$ by an irreducible curve C to obtain a family of cycles $\{Z(t)\}_{t \in C}$ and a normal function $\nu : C \to \mathcal{J}^{m-1}(X_C/C)$ defined by $\nu(t) = \mathrm{AJ}(Z(t))$. As $\nu(t_0) \neq 0$, it follows that $\nu(t) \neq 0$ for general t.

For the second statement, choose $\lambda(t_0) \in \Delta_{1,1}(t_0)$ such that $\nu(t_0) \notin J^{m-1}(X(t_0))_{\mathrm{tors}}$. By specialisation to $\lambda(t_0)$; we see that the set

$$T_n = \{t \in T \mid \nu(t) \in J^{m-1}(X(t))_n\}$$

is a proper Zariski closed subset of T for every $n \in \mathbb{N}$. Hence the statement follows. $\square$

4.4 Regulator maps

This section is devoted to the second and third case mentioned in Section 4.2. We shall concentrate on the second case, complete intersections of three quadrics, the third case being similar but easier.

Our aim is to prove the following result [53],[54].

Theorem 4.4.1 *Let $X \subset \mathbb{P}^{2m+1}$ be a smooth complete intersection of three quadrics. Modulo torsion, the image of the map*

$$c_{m,1} : \mathrm{CH}^m(X,1) \to H^{2m-1}_{\mathcal{D}}(X,\mathbb{Z}(m))$$

is contained in $i^* H^{2m-1}_{\mathcal{D}}(\mathbb{P}^{2m+1},\mathbb{Z}(m)) \cong \mathbb{C}^*$ if X is very general and $m \geq 3$.

The quadrics defining X correspond to an element of the vector space

$$V = \oplus^3 H^0(\mathbb{P}^{2m+1}, \mathcal{O}_{\mathbb{P}}(2)).$$

Standard techniques [51] show that to prove the theorem, it suffices to find a Zariski open subset $U \subset \mathbb{P}(V)$ such that the pullback $\varphi_T : X_T \to T$ of the universal family satisfies $H^{2m-1}(\mathbb{P}^{2m+1} \times T, X_T, \mathbb{Q}) = 0$ for every smooth morphism $T \to U$. The latter statement is equivalent to the vanishing of $H^1(T, (R^{2m-2}\varphi_{T,*}\mathbb{Q})_v)$. By the Cayley trick, this is equivalent to the vanishing of $H^1(T, (R^{2m+2} f_{T,*}\mathbb{Q})_v)$ for the associated family of quadric bundles $f_T : \mathcal{X}_T \to T$.

Put $W = H^0(\mathbb{P}^{2m+2}, \mathcal{O}_{\mathbb{P}}(2))$ and consider the map

$$H : V \to W, \quad H(Q_0, Q_1, Q_2) = \det(\lambda_0 Q_0 + \lambda_1 Q_1 + \lambda_2 Q_2)$$

that sends a net of quadrics to the equation of the discriminant curve of the associated quadric bundle.

Theorem 4.4.2 (Dixon) *The rational map $h : \mathbb{P}(V) - - > \mathbb{P}(W)$ induced by H is dominant.*

Dixon's proof appeared in [32]; see [10, Prop. 4.2 and Remark 4.4] for a modern proof.

In the sequel we write $\mathbb{P}(V)_{\mathrm{ns}}$ (resp. $\mathbb{P}(W)_{\mathrm{ns}}$) for the complement of the discriminant locus in $\mathbb{P}(V)$ (resp. $\mathbb{P}(W)$). By [40, III, Lemma 10.5], Dixon's theorem implies that there exists a Zariski open subset $U \subset \mathbb{P}(V)$ such that $h : U \to \mathbb{P}(W)$ is smooth. Shrinking U if necessary, we may assume that $U \subset \mathbb{P}(V)_{\mathrm{ns}}$ and $h(U) \subset \mathbb{P}(W)_{\mathrm{ns}}$. Consider the universal family $S_W \to \mathbb{P}(W)$ whose fiber over $[F]$ is the double covering of $\mathbb{P}^2$ ramified along $V(F)$, and let S_T be its pullback to T with projection $g_T : S_T \to \mathbb{P}^2 \times T$. The homomorphism

$$\Gamma'_{T,*} : (R^2 g_{T,*}\mathbb{Q})_v \to (R^{2m+2} f_{T,*}\mathbb{Q})_v$$

induced by the universal family $\Gamma_T \to F_m(\mathcal{X}_T/T)$ is an isomorphism by the proper base change theorem and Lamma 3.1.7. The Cayley trick implies that the correspondence

$$\begin{array}{ccc} X_T \times \mathbb{P}^2_T & \hookrightarrow & \mathcal{X}_T \\ \downarrow & & \\ X_T & & \end{array}$$

induces an isomorphism $(R^{2m+2}f_{T,*}\mathbb{Q})_v \cong (R^{2m-2}\varphi_{T,*}\mathbb{Q})_v$. Combining these two isomorphisms, we obtain

$$(R^2 g_{T,*}\mathbb{Q})_v \cong (R^{2m-2}\varphi_{T,*}\mathbb{Q})_v. \tag{4.1}$$

To state the following result, we need some notation. Given a continuous map $f : X \to Y$ of topological spaces, let $M(f)$ be the mapping cylinder of f and put

$$H^k(Y, X; \mathbb{Z}) = H^k(M(f), \mathbb{Z}).$$

These groups fit into a long exact sequence

$$\ldots \to H^k(Y) \xrightarrow{f^*} H^k(X) \to H^k(Y, X) \to H^{k+1}(X) \to \ldots$$

of cohomology groups; if f is an inclusion, we recover the usual relative cohomology of the pair (X, Y).

Theorem 4.4.3 *With the above notation, we have isomorphisms*

$$H^{k+2}(\mathbb{P}^2_T, S_T) \cong H^{2m+k-2}(\mathbb{P}^{2m+1}_T, X_T)$$

for all $k \geq 0$.

Proof: This follows from (4.1), since

$$\begin{aligned} H^{k+2}(\mathbb{P}^2_T, S_T) &\cong H^{k-1}(T, (R^2 g_{T,*}\mathbb{Q})_v) \\ H^{2m+k-2}(\mathbb{P}^{2m+1}_T, X_T) &\cong H^{k-1}(T, (R^{2m-2}\varphi_{T,*}\mathbb{Q})_v). \end{aligned}$$

□

A straightforward generalisation of the infinitesimal computations in [50] gives a version of Nori's connectivity theorem for cyclic coverings; see [53]. We only need the following special case of this result.

Theorem 4.4.4 *Let* $T \to \mathbb{P}(W)_{\mathrm{ns}}$ *be a smooth morphism. If* $m \geq 3$ *then* $H^k(\mathbb{P}^2_T, S_T, \mathbb{Q}) = 0$ *for all* $k \leq 3$.

Corollary 4.4.5 *With the above notation, we have* $H^q(\mathbb{P}^{2m+1}_T, X_T, \mathbb{Q}) = 0$ *for all* $q \leq 2m - 1$.

Proof: For $q \leq 2m - 3$ the statement follows from the Lefschetz hyperplane theorem and the Leray spectral sequence. For $q \in \{2m-2, 2M-1\}$ we apply Theorems 4.4.3 and 4.4.4. □

A standard argument now finishes the proof of Theorem 4.4.1; see e.g. [51].

Remark 4.4.6 The same argument works for $r = 1$. If $r \geq 3$ the techniques of this section do not apply, since a dimension count shows that the map

$$H : \bigoplus^{r+1} H^0(\mathbb{P}^{2m+1}, \mathcal{O}_{\mathbb{P}}(2)) \to H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}}(2m+2))$$

defined by $H(Q_0, \ldots, Q_r) = \det(\lambda_0 Q_0 + \ldots + \lambda_r Q_r)$ is not surjective.

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