Lectures on Nori's connectivity theorem

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The aim of these lectures is to discuss Nori's connectivity theorem and its applications to the theory of algebraic cycles. I have tried to clarify some of the underlying ideas by emphasizing the relationship of Nori's theorem with the theorems of Griffiths and Green–Voisin on the image of the Abel–Jacobi map for hypersurfaces in projective space.

The notes are divided into 6 sections that roughly correspond to my 6 lectures at the summer school in Grenoble. I have added a short Appendix on Deligne cohomology.

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1 Normal functions

Let X be a smooth projective variety over $\mathbb{C}$. The main invariants used to study the Chow group $\mathrm{CH}^p(X)$ of codimension p cycles on X are the *cycle class map*

$$\mathrm{cl}_X^p : \mathrm{CH}^p(X) \to H^{2p}(X, \mathbb{Z})$$

and the *Abel–Jacobi map*

$$\psi_X^p : \mathrm{CH}^p_{\mathrm{hom}}(X) \to J^p(X),$$

which is defined on the kernel of cl_X^p; its target is the intermediate Jacobian

$$\begin{aligned} J^p(X) &= H^{2p-1}(X, \mathbb{C})/F^p H^{2p-1}(X, \mathbb{C}) + H^{2p-1}(X, \mathbb{Z})/\mathrm{tors} \\ &\cong F^{n-p+1} H^{2n-2p+1}(X, \mathbb{C})^\vee / H_{2n-2p+1}(X, \mathbb{Z}). \end{aligned}$$

The Hodge conjecture gives a conjectural description of the image of cl_X^p. Little is known about the kernel and the image of ψ_X^p, except in special cases such as curves and Fano threefolds. A natural idea is to consider a family of smooth projective varieties $\{X_s\}_{s \in S}$ and to study holomorphic sections of the fiber space of intermediate Jacobians over S to obtain information about cycles on the general fiber. Such sections are called normal functions; they were introduced by Poincaré for families of curves on algebraic surfaces and by Griffiths for algebraic cycles on higher–dimensional varieties.

Let $f : X \to S$ be a smooth projective morphism of quasi–projective varieties. Suppose that the fibers $X_s = f^{-1}(s)$ have dimension $2m-1$. The intermediate Jacobians $J^m(X_s)$ of the fibers fit together to give a holomorphic fiber space of complex tori

$$J^m(X/S) = \cup_{s \in S} J^m(X_s).$$

The sheaf $H_{\mathbb{Z}}^{2m-1} = R^{2m-1} f_* \mathbb{Z}/\mathrm{tors}$ is a local system of abelian groups. The associated Hodge bundle $\mathcal{H}^{2m-1} = H_{\mathbb{Z}}^{2m-1} \otimes_{\mathbb{Z}} \mathcal{O}_S$ is a vector bundle that carries a flat connection ∇, the Gauss–Manin connection. The Hodge bundle is filtered by holomorphic subbundles $\mathcal{F}^p$. The sheaf

$$\mathcal{J}^m = \mathcal{H}^{2m-1}/\mathcal{F}^m + H_{\mathbb{Z}}^{2m-1}$$

is the sheaf of sections of the fibration $\pi : J^m(X/S) \to S$.

Definition 1.1 *A normal function is a holomorphic section $\nu \in H^0(S, \mathcal{J}^m)$.*

The exact sequence of sheaves

$$0 \to H_{\mathbb{Z}}^{2m-1} \to \mathcal{H}^{2m-1}/\mathcal{F}^m \to \mathcal{J}^m \to 0$$

induces a map $\partial : H^0(S, \mathcal{J}^m) \to H^1(S, H_{\mathbb{Z}}^{2m-1})$ that associates to a normal function ν its *cohomological invariant* $\partial(\nu) \in H^1(S, H_{\mathbb{Z}}^{2m-1})$. The Gauss–Manin connection $\nabla : \mathcal{H}^{2m-1} \to \Omega_S^1 \otimes \mathcal{H}^{2m-1}$ restricts to a map

$$\nabla : \mathcal{F}^m \to \Omega_S^1 \otimes \mathcal{F}^{m-1}$$

by Griffiths transversality. As $\nabla(H_{\mathbb{Z}}^{2m-1}) = 0$ we obtain an induced map

$$\overline{\nabla} : \mathcal{J}^m \to \Omega_S^1 \otimes \mathcal{H}^{2m-1}/\mathcal{F}^{m-1}$$

whose kernel is denoted by $\mathcal{J}_h^m$. The holomorphic sections of this sheaf are called *quasi–horizontal* normal functions.

Definition 1.2 *Let $f : X \to S$ be a smooth projective morphism. The group $Z^p(X/S)$ of relative codimension p cycles over S is the free abelian group generated by irreducible subvarieties $Z \subset X$ that are flat of relative dimension* $\dim X - \dim S - p$ *over S.*

We write $Z^m_{\mathrm{hom}}(X/S)$ for the subgroup of relative codimension m cycles whose restriction to every fiber X_s is homologically equivalent to zero. Given $Z \in Z^m_{\mathrm{hom}}(X/S)$, it is possible to choose a family of $(2m-1)$–chains $\gamma = (\gamma_s)$ such that $\partial\gamma_s = Z_s$ for all $s \in S$. We then obtain a C^∞ section $\tilde{\nu} \in \Gamma(S, (\mathcal{F}^m)^\vee)$ by setting

$$\langle \tilde{\nu}(s), \omega_s \rangle = \int_{\gamma_s} \omega_s$$

for a C^∞ section $\omega = (\omega_s) \in \Gamma(S, \mathcal{F}^m)$. By projection to $\mathcal{J}^m$ we obtain a section $\nu_Z \in \Gamma(S, \mathcal{J}^m)$.

Let ξ be a vector field on X. There exists a *contraction map*

$$i_\xi : A^k(X) \to A^{k-1}(X)$$

that is defined as the unique derivation that equals the evaluation map on 1–forms and is linear for C^∞ functions. Let ξ be a local lifting of the vector field

$\frac{\partial}{\partial s}$. Locally, the flow associated to ξ defines diffeomorphisms $\varphi_s : X_0 \to X_s$ (Ehresmann's fibration theorem). The *Lie derivative* of a family $\omega = (\omega_s)$ of differential k–forms with respect to ξ is defined by

$$L_\xi \omega = \frac{\partial}{\partial s}|_{s=0} \varphi_s^* \omega_s.$$

A similar formula defines $L_{\bar{\xi}}\omega$ if $\bar{\xi}$ is a local lifting of $\frac{\partial}{\partial \bar{s}}$. The Lie derivative can be computed using the *Cartan formula*

$$L_\xi \omega = d i_\xi \Omega + i_\xi d\Omega$$

where Ω is a k–form on X such that $\Omega|_{X_s} = \omega_s$ for all $s \in S$. By definition, the Gauss–Manin derivative along ξ of a family of closed forms ω is given by

$$\nabla_\xi \omega = [L_\xi \omega].$$

This is well–defined and maps families of closed forms to families of closed forms since $[L_\xi, d] = 0$ by the Cartan formula.

Proposition 1.3 *The section $\nu = \nu_Z \in \Gamma(S, \mathcal{J}^m)$ associated to a relative cycle $Z \in Z^m_{\mathrm{hom}}(X/S)$ is a quasi–horizontal normal function.*

Proof: It suffices to cover S by contractible open subsets and to verify the assertion locally. Hence we may assume that S is a disc. Let $\mathcal{A}^1(X/S) = \mathcal{A}^1(X)/f^*\mathcal{A}^1(S)$ be the sheaf of relative differentiable 1–forms. The sheaf $\mathcal{A}^{2m-1}(X/S) = \bigwedge^{2m-1} \mathcal{A}^1(X/S)$ admits a decomposition

$$\mathcal{A}^{2m-1}(X/S) = \bigoplus\nolimits_{p+q=2m-1} \mathcal{A}^{p,q}(X/S)$$

Set $\mathcal{F}^m\mathcal{A}^{2m-1}(X/S) = \oplus_{p \geq m} \mathcal{A}^{p,2m-1-p}(X/S)$, and let $\omega = (\omega_s)_{s\in S}$ be a C^∞ section of $\mathcal{F}^m\mathcal{A}^{2m-1}(X/S)$ such that ω_s is closed for all $s \in S$. There exists $\Omega \in F^m A^{2m-1}(X)$ such that $\Omega|_{X_s} = \omega_s$ for all $s \in S$. The form Ω is uniquely determined if we impose the condition that $i_\chi \Omega = 0$ for every horizontal vector field χ, i.e., a vector field of the form f^*v, $v \in \Gamma(S, T_S)$.

By Ehresmann's fibration theorem there is a diffeomorphism $X \cong X_0 \times S$ that induces diffeomorphisms $\varphi_s : X_0 \to X_s$ for all $s \in S$. Choose $\gamma_0 \in C_{2m-2}(X_0)$ such that $\partial\gamma_0 = Z_0$ and define $\gamma_s = (\varphi_s)_*\gamma_0$. Set $h(s) = \langle \tilde{\nu}(s), \omega_s \rangle$.

Let ξ be a lifting of the vector field $\frac{\partial}{\partial s}$, and let $\bar{\xi}$ be a lifting of $\frac{\partial}{\partial \bar{s}}$. To show that h is holomorphic we compute

$$\begin{aligned} \frac{\partial}{\partial \bar{s}}|_{s=0} h(s) &= \frac{\partial}{\partial \bar{s}}|_{s=0} \int_{\varphi_{s*}\gamma_0} \omega_s = \frac{\partial}{\partial \bar{s}}|_{s=0} \int_{\gamma_0} \varphi_s^* \omega_s = \int_{\gamma_0} \frac{\partial}{\partial \bar{s}}|_{s=0} \varphi_s^* \omega_s \\ &= \int_{\gamma_0} L_{\bar{\xi}} \omega|_{X_0}. \end{aligned}$$

Using the Cartan formula we find that $L_{\bar{\xi}}\omega = i_{\bar{\xi}} d\Omega$ (remember that $i_{\bar{\xi}}\Omega = 0$). Since contraction with $\bar{\xi}$ cannot annihilate any $dz's$ we find that $L_{\bar{\xi}}\omega|_{X_0} \in F^m A^{2m-1}(X_0)$. If ω is a holomorphic section of $\mathcal{F}^m$ then $\nabla_{\bar{\xi}}\omega = 0$, hence $L_{\bar{\xi}}\omega$ is exact. An improtant consequence of Hodge theory is that the differential $d : A^k(X) \to A^{k+1}(X)$ is *strictly compatible* with $F^\bullet$. This means that we can find $\eta \in F^m A^{2m-2}(X_0)$ such that $L_{\bar{\xi}}\omega = d\eta$. Using the Stokes formula we find that

$$\frac{\partial}{\partial \bar{s}}|_{s=0} h(s) = \int_{Z_0} \eta = 0$$

since the complex dimension of Z_0 is $m-1$. This proves that ν is a holomorphic section of $\mathcal{J}^m$.

To check the quasi–horizontality property, we use the Leibniz rule

$$\frac{\partial}{\partial s}|_{s=0} h(s) = \langle \nabla_\xi \tilde{\nu}(0), \omega_0 \rangle + \langle \tilde{\nu}(0), \nabla_\xi \omega(0) \rangle.$$

If we can show that

$$(*) \qquad \langle \nabla_\xi \tilde{\nu}(0), \omega_0 \rangle = 0$$

for every section $\omega \in \Gamma(S, \mathcal{F}^{m+1})$ it follows that

$$\nabla_\xi \tilde{\nu}(0) \in F^{m+1} H^{2m-1}(X_0)^\perp = F^{m-1} H^{2m-1}(X_0)$$

which means that $\overline{\nabla}\nu(0) = 0$. To verify $(*)$ we compute

$$\frac{\partial}{\partial s}|_{s=0} h(s) = \int_{\gamma_0} L_\xi \omega|_{X_0}.$$

As the closed form $L_\xi \omega|_{X_0} = i_\xi d\Omega|_{X_0} \in F^m A^{2m-1}(X_0)$ represents $\nabla_\xi \omega$ we have

$$\nabla_\xi \omega = i_\xi d\Omega|_{X_0} + d\eta$$

where η can be chosen in $F^m A^{2m-1}(X_0)$, again by strict compatibility of d with the Hodge filtration. The Stokes formula then shows that

$$\int_{\gamma_0} d\eta = 0,$$

hence

$$\frac{\partial}{\partial s}|_{s=0} h(s) = \int_{\gamma_0} i_\xi d\Omega|_{X_0} = \langle \tilde{\nu}(0), \nabla_\xi \omega(0) \rangle$$

and this is equality is equivalent to $(*)$. $\square$

Let Y be a smooth projective variety of even dimension $2m$ and let $\{X_t\}_{t\in T}$ be a family of smooth hypersurface sections of Y. Let X_T be the total space of the family $\{X_t\}$ with inclusion $r : X_T \to Y \times T$. Suppose there exists a codimension m cycle Z on Y such that $Z \cap X_t$ is homologically equivalent to zero for all $t \in T$. The normal function associated to the relative cycle $r^*(Z \times T) \in Z^m_{\mathrm{hom}}(X_T/T)$ is denoted by ν_Z. By construction $\nu_Z(t)$ is the image of Z_t under the Abel–Jacobi map on X_t.

Using Deligne cohomology (see Appendix A) it is possible to associate a normal function to Hodge classes on Y. Define

$$\mathrm{Hdg}^m(Y)_0 = \ker(i^* : \mathrm{Hdg}^m(Y) \to \mathrm{Hdg}^m(X))$$

and consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & J^m(Y) & \to & H^{2m}_{\mathcal{D}}(Y, \mathbb{Z}(m)) & \to & \mathrm{Hdg}^m(Y) & \to & 0 \\ & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} & & \\ 0 & \to & J^m(X) & \to & H^{2m}_{\mathcal{D}}(X, \mathbb{Z}(m)) & \to & \mathrm{Hdg}^m(X) & \to & 0. \end{array}$$

Given $\xi \in \mathrm{Hdg}^m(Y)_0$, choose a lifting $\tilde{\xi} \in H^{2m}_{\mathcal{D}}(Y, \mathbb{Z}(m))$. As $i^*\tilde{\xi}$ maps to zero in $\mathrm{Hdg}^m(X)$, it belongs to $J^m(X)$. To get a well–defined map we have to pass to the quotient

$$J^m_{\mathrm{var}}(X) = J^m(X)/i^* J^m(Y).$$

Define $\nu(s) = i_s^* \tilde{\xi}$. One can show that $\nu \in H^0(S, \mathcal{J}^m_{\mathrm{var}})$ is a quasi–horizontal normal function (cf. [12, Lecture 6]).

Examples 1.4 Some other examples of normal functions:

(i) Let C be a non–hyperelliptic curve of genus 4. It is known that the canonical image $\varphi_K(C) \subset \mathbb{P}^3$ is the complete intersection of a quadric Q and a cubic F. Let ℓ_1, ℓ_2 be two lines from the different rulings of the quadric and set $D = \ell_1 \cap F - \ell_2 \cap F \in Z^1_{\text{hom}}(C)$. If $C_S = \{C_s\}_{s \in S}$ is a family of non–hyperelliptic genus 4 curves, we obtain a relative cycle $D_S = \cup_{s \in S} D_s \in Z^1_{\text{hom}}(C_S/S)$ and a normal function ν_D.

(ii) Let C be a smooth curve of genus $g \geq 3$. The choice of a base point $x \in C$ defines an embedding $i_x : C \to J(C)$ of C into the g–dimensional abelian variety $J(C)$. There is an involution i on $J(C)$ given by multiplication by -1. Define $C_x^+ = i_x(C)$, $C_x^- = i_*(C_x^+)$ and $Z_{C,x} = C_x^+ - C_x^-$. As i_* acts as the identity on $H^{2g-2}(J(C), \mathbb{Z}) = \bigwedge^{2g-2} H^1(J(C), \mathbb{Z})$, we have $[Z_{C,x}] \in \mathrm{CH}^{g-1}_{\text{hom}}(J(C))$. We obtain a normal function $\tilde{\nu}$ over an open subset of the moduli space $M_{g,1}$ of pointed genus g curves associated to the relative cycle $Z = \cup_{(C,x)} Z_{C,x}$. Let $PH_3(J(C), \mathbb{Z})$ be the cokernel of the map

$$H_1(J(C), \mathbb{Z}) \to H_3(J(C), \mathbb{Z})$$

given by Pontryagin product with $[C_x] \in H_2(J(C), \mathbb{Z})$. One can verify that the projection of the Abel–Jacobi image of $C_x^+ - C_x^-$ to the primitive intermediate Jacobian

$$J^{g-1}_{\text{pr}}(J(C)) = F^2 H^3_{\text{pr}}(J(C), \mathbb{C})^\vee / PH_3(J(C), \mathbb{Z})$$

does not depend on the choice of the base point x. Hence $\tilde{\nu}$ descends to a section $\nu \in H^0(U, \mathcal{J}^{g-1}_{\text{pr}})$ defined over a Zariski open subset $U \subset M_g$.

A *local lifting* of a normal function ν is a local section of the Hodge bundle that projects to ν. For normal functions that satisfy the quasi–horizontality property we can define a new invariant that measures the obstruction for the existence of *flat* local liftings. To define this invariant, consider the de Rham complex for the Hodge bundle $\mathcal{H} = \mathcal{H}^{2m-1}$

$$\Omega^\bullet(\mathcal{H}) = (\mathcal{H} \xrightarrow{\nabla} \Omega^1_S \otimes \mathcal{H} \xrightarrow{\nabla} \Omega^2_S \otimes \mathcal{H} \to \dots)$$

and its subcomplex

$$\Omega^\bullet(\mathcal{F}^m) = (\mathcal{F}^m \xrightarrow{\nabla} \Omega^1_S \otimes \mathcal{F}^{m-1} \xrightarrow{\nabla} \Omega^2_S \otimes \mathcal{F}^{m-2} \to \dots).$$

Let $\tilde{\Omega}^\bullet(\mathcal{F}^m) = \Omega^\bullet(\mathcal{F}^m) \oplus H_{\mathbb{Z}}$ be the complex obtained by adding the local system $H_{\mathbb{Z}}$ in degree zero. The quotient of the de Rham complex by this modified subcomplex is the complex

$$\Omega^\bullet(\mathcal{H}/\mathcal{F}^m) = (\mathcal{J}^m \xrightarrow{\overline{\nabla}} \Omega^1_S \otimes \mathcal{H}/\mathcal{F}^{m-1} \to \ldots).$$

The connecting homomorphism

$$\mathcal{J}^m_h = \mathcal{H}^0(\Omega^\bullet(\mathcal{H}/\mathcal{F}^m)) \to \mathcal{H}^1(\tilde{\Omega}^\bullet(\mathcal{F}^m)) = \mathcal{H}^1(\Omega^\bullet(\mathcal{F}^m))$$

induces a map

$$\delta : H^0(S, \mathcal{J}^m_h) \to H^0(S, \mathcal{H}^1(\Omega^\bullet(\mathcal{F}^m))).$$

We call $\delta\nu$ the *infinitesimal invariant* of ν. It is obtained as follows: choose an open covering $\mathcal{U} = \{U_\alpha\}$ of S and apply ∇ to a local lifting $\tilde{\nu}_\alpha$ of ν_α; by quasi–horizontality, $\nabla\tilde{\nu}_\alpha$ comes from a local section of $\Omega^1_S \otimes \mathcal{F}^{m-1}$, which is annihilated by ∇ and is well defined modulo sections in the image of $\nabla : \mathcal{F}^m \to \Omega^1_S \otimes \mathcal{F}^{m-1}$. The local sections $\delta\nu_\alpha = [\nabla\tilde{\nu}_\alpha]$ patch together to give a global section $\delta\nu \in \Gamma(S, \mathcal{H}^1(\Omega^\bullet(\mathcal{F}^m)))$.

Lemma 1.5

$$\delta\nu = 0 \Longleftrightarrow \nu \textit{ has flat local liftings.}$$

Proof: If $\delta\nu = 0$ then there exists locally a section f of $\mathcal{F}^m$ such that $\nabla\tilde{\nu} = \nabla f$, hence $\tilde{\nu} - f = \nabla(\lambda)$ where λ is locally constant and $\hat{\nu} = \tilde{\nu} - f$ is a flat local lifting of ν. The other direction of the equivalence is clear. □

The complex $\Omega^\bullet(\mathcal{F}^m)$ is filtered by subcomplexes $F^p\Omega^\bullet(\mathcal{F}^m) = \Omega^\bullet(\mathcal{F}^p)$ $(p \geq m)$ with graded pieces

$$\mathrm{Gr}^p_F\, \Omega^\bullet(\mathcal{F}^m) = (\mathcal{H}^{p,m-p-1} \xrightarrow{\overline{\nabla}} \mathcal{H}^{p-1,m-p} \to \ldots).$$

Note that the differential $\overline{\nabla}$ in these complexes is $\mathcal{O}_S$–linear. There is a natural map

$$H^0(S, \mathcal{H}^1(\Omega^\bullet(\mathcal{F}^m))) \to H^0(S, \mathcal{H}^1(\mathrm{Gr}^m_F\, \Omega^\bullet(\mathcal{F}^m))).$$

The image $\delta_1\nu$ of $\delta\nu$ under this map is the infinitesimal invariant of normal functions defined by Griffiths.

Remark 1.6 (i) If $\{Z(t)\}_{t \in T}$ is a family of codimension m cycles on X, we obtain a relative cycle on the trivial family $X \times T$. In this case, the sheaf $\mathcal{J}^m$ is the constant sheaf $J^m(X)$ and the associated normal function is a map $\nu : T \to J^m(X)$, defined after the choice of a base point t_0 by $\nu(t) = \psi(Z(t) - Z(t_0))$. The infinitesimal invariant $\delta_1 \nu$ is an element of

$$\Omega^1_T \otimes H^{m-1,m}(X) \cong \mathrm{Hom}(T, H^{m-1,m}(X)).$$

It coincides with the differential ν_* of ν, which is called the infinitesimal Abel–Jacobi map.

(ii) The infinitesimal invariant $\delta_1 \nu$ often carries geometric information. For genus 4 curves, Griffiths [15] showed that it determines the cubic containing the canonical image of the curve. For genus 3 curves, Collino and Pirola [7] showed that it determines the canonical equation of the curve.

Remark 1.7 Griffiths defined the *fixed part* of $\mathcal{J}^m$ as the sheaf

$$\mathcal{J}^m_{\text{fix}} = H_{\mathbb{C}}/\mathcal{F}^m \cap H_{\mathbb{C}} + H_{\mathbb{Z}}.$$

The reason for this terminology is that if $\{X_s\}$ is a family of hypersurface sections of Y, the fixed part can be identified with the constant sheaf $J^m(Y)$ using a monodromy argument. (We shall see this in the third lecture for $Y = \mathbb{P}^{2m}$.) We have an exact sequence

$$0 \to \mathcal{J}^m_{\text{fix}} \to \mathcal{J}^m_h \to \mathcal{H}^1(\Omega^\bullet(\mathcal{F}^m)) \to 0,$$

hence

$$\delta\nu = 0 \Longleftrightarrow \nu \in H^0(S, \mathcal{J}^m_{\text{fix}}).$$

To conclude this lecture, we mention without proof two important theorems on normal functions. Suppose that S is a smooth curve and $f : X \to S$ admits a compactification $\bar{f} : \bar{X} \to \bar{S}$ such that the fibers $\bar{f}^{-1}(s)$ over the points $s \in \bar{S} \backslash S$ are divisors with simple normal crossings. By work of Schmid and Steenbrink it is possible to extend the Hodge bundle and its subbundles to vector bundles $\bar{\mathcal{H}}$ and $\bar{\mathcal{F}}^m$ on $\bar{S}$. Let $j : S \to \bar{S}$ be the inclusion map. The *Zucker extension* $\bar{\mathcal{J}}^m$ of $\mathcal{J}^m$ is the sheaf

$$\bar{\mathcal{J}}^m = \bar{\mathcal{H}}/\bar{\mathcal{F}}^m + j_* H_{\mathbb{Z}}.$$

Strictly speaking, normal functions should be defined as global sections of $\bar{\mathcal{J}}^m$ (i.e., sections of $\mathcal{J}^m$ that extend over the singular fibers). There exists a map

$$\partial : H^0(\bar{S}, \bar{\mathcal{J}}^m) \to H^1(\bar{S}, j_* H_{\mathbb{Q}}).$$

Theorem 1.8 (Zucker) *The group $H^1(\bar{S}, j_* H_{\mathbb{Q}})$ carries a Hodge structure of weight $2m$ and the image of ∂ coincides with the set of Hodge classes* $\mathrm{Hdg}^m\, H^1(\bar{S}, j_* H_{\mathbb{Q}})$.

There is also a criterion for extendability of normal functions. For every $s \in \bar{S} \setminus S$, let $\Delta^*(s)$ be a punctured disc centered at s and let $\delta_s(\nu) \in H^1(\Delta^*(s), H_{\mathbb{Z}})$ be the cohomological invariant of $\nu|_{\Delta^*(s)}$.

Theorem 1.9 (El Zein–Zucker) *Let ν be a normal function. If $\delta_s(\nu) = 0$ for all $s \in \bar{S} \setminus S$, then ν extends to a section $\bar{\nu} \in H^0(\bar{S}, \bar{\mathcal{J}}^m)$.*

Bibliographical hints. A good introduction to normal functions is Zucker's paper [32]; see also [14]. The proof of Proposition 1.3 is taken from [26]. Zucker's theorem on normal functions can be found in [30] and [31]. For a discussion of extendability of normal functions, see [8]. The invariant $\delta_1\nu$ was discovered by Griffiths [15]. The definition of $\delta\nu$ is due to Green [11].

2 Griffiths's theorem

Let X be a smooth projective variety. The group $\mathrm{CH}^p_{\mathrm{hom}}(X)$ of codimension p cycles homologically equivalent to zero contains as a subgroup the group $\mathrm{CH}^p_{\mathrm{alg}}(X)$ of cycles algebraically equivalent to zero. For divisors ($p = 1$) and zero–cycles ($p = \dim X$) both groups coincide but in general they may be different. The quotient group

$$\mathrm{Griff}^p(X) = \mathrm{CH}^p_{\mathrm{hom}}(X)/\mathrm{CH}^p_{\mathrm{alg}}(X)$$

is called the *Griffiths group* of codimension p cycles.

In 1969 Griffiths showed that there exist quintic threefolds $X \subset \mathbb{P}^4$ such that the difference of two lines on X is not algebraically equivalent to zero. This follows from the theorem below. Recall that a property (P) is said to hold for a *very general* point of a topological space T if the subset of elements that do not satsify (P) is a countable union of proper closed subsets of T.

Theorem 2.1 (Griffiths) *Let Y be a smooth projective variety of even dimension $2m$ and let $\{X_t\}_{t\in\mathbb{P}^1}$ be a Lefschetz pencil of hyperplane sections of Y. Suppose that $H^{2m-1}(Y) = 0$ and that*

$$H^{2m-1}(X_t, \mathbb{C}) \neq H^{m,m-1}(X_t) \oplus H^{m-1,m}(X_t) \tag{1}$$

If $Z \in Z^m(Y)$ and $Z \cap X_t$ is algebraically equivalent to zero for very general $t \in \mathbb{P}^1$, then Z is homologically equivalent to zero.

Remark 2.2 The assumption $H^{2m-1}(Y) = 0$ is only included to simplify the proof; it can be omitted. In a later lecture we shall explain Nori's generalisation of Theorem 2.1.

Before we start with the proof of Theorem 2.1 we introduce some notation: let $B = X_0 \cap X_\infty$ be the base locus of the pencil, and let $\tilde{Y}$ be the blow–up of Y along B. Let U be the complement of the discriminant locus $\Delta \subset \mathbb{P}^1$. We have a diagram

$$\begin{array}{ccccc} X_U & \xrightarrow{r} & \tilde{Y} & \xrightarrow{\pi} & Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}} & & \\ U & \xrightarrow{j} & \mathbb{P}^1. & & \end{array}$$

To prove Theorem 2.1 we need several lemmas. Let

$$J^m_{\mathrm{alg}}(X) = \mathrm{im}(\psi^m_X : \mathrm{CH}^m_{\mathrm{alg}}(X) \to J^m(X))$$

be the algebraic part of the intermediate Jacobian of $X = X_t$.

Lemma 2.3 *With the hypotheses of Theorem 2.1 we have $J^m_{\mathrm{alg}}(X_t) = 0$ if $t \in U$ is very general.*

Proof: Let $X = X_t$ be a general hyperplane section of Y. Recall that a cycle $z \in Z^m(X)$ is algebraically equivalent to zero if there exist a variety S, a relative cycle $\mathcal{Z} \in Z^m(X \times S/S)$ and two points s_0, s_1 in S such that $z = \mathcal{Z}(s_0) - \mathcal{Z}(s_1)$. We may assume that S is a smooth irreducible curve. Define a map

$$g : S \to J^m(X)$$

by $g(s) = \psi_X(\mathcal{Z}(s_0) - \mathcal{Z}(s))$. As $\psi_X(z) = g(s_1)$ it suffices to study the image of g. As g is holomorphic, it factorises over a map

$$h : J(S) \to J^m(X)$$

by the universal property of the Jacobian. By Poincaré duality, the induced map on homology groups

$$h_* : H_1(J(S), \mathbb{Z}) = H_1(S, \mathbb{Z}) \to H_1(J^m(X), \mathbb{Z}) = H_{2m-1}(X, \mathbb{Z})$$

gives a map

$$h_* : H^1(S) \to H^{2m-1}(X).$$

This map is a morphism of Hodge structures of type $(m-1, m-1)$ induced by the corrspondence $[\mathcal{Z}] \in \mathrm{CH}^m(X \times S)$. As it induces the map h by passage to the quotient, we find that $J^m_{\mathrm{alg}}(X)$ is the intermediate Jacobian associated to a sub–Hodge structure

$$H_{\mathrm{alg}} \subset H^{m-1,m}(X) \oplus H^{m,m-1}(X) \cap H^{2m-1}(X, \mathbb{Q}).$$

Let

$$\rho : \pi_1(U) \to \mathrm{Aut}\, H^{2m-1}(X, \mathbb{Q})$$

be the monodromy representation, and let $\Gamma = \mathrm{im}\,\rho$ be the mondromy group. By Picard–Lefschetz theory we know that $H^{2m-1}(X, \mathbb{Q})$ is an irreducible Γ–module. If $t \in U$ is very general, it is possible to 'spread out' every cycle on X_t to a relative cycle over U (this will be explained in more detail in the next lecture), hence $H_{\mathrm{alg}} \subset H^{2m-1}(X, \mathbb{Q})$ is a Γ–submodule. If $H_{\mathrm{alg}} \neq 0$ we would get $H_{\mathrm{alg}} = H^{2m-1}(X, \mathbb{Q})$, which is impossible by condition (1) of Theorem 2.1. $\square$

The Leray spectral sequence for the map $f : X_U \to U$ defines a filtration $L^\bullet$ on $H^{2m}(X_U)$. We have

$$L^1 H^{2m}(X_U, \mathbb{Q}) = \ker(H^{2m}(X_U, \mathbb{Q}) \to H^0(U, R^{2m} f_* \mathbb{Q})).$$

Note that the primitive cohomology

$$H^{2m}_{\mathrm{pr}}(Y,\mathbb{Q}) = \ker(\cup c_1(\mathcal{O}_Y(1)) : H^{2m}(Y,\mathbb{Q}) \to H^{2m+2}(Y,\mathbb{Q}))$$

coincides with the kernel of the restriction map $i^* : H^{2m}(Y,\mathbb{Q}) \to H^{2m}(X_t,\mathbb{Q})$ if X_t is smooth. Hence, if $\alpha \in H^{2m}_{\mathrm{pr}}(Y,\mathbb{Q})$ then $\pi^*\alpha \in L^1H^{2m}(X_U,\mathbb{Q})$. As $f : X_U \to U$ is a smooth morphism, the sheaf $R^{2m-1}f_*\mathbb{Q}$ is locally constant; we denote it by $H^{2m-1}_{\mathbb{Q}}$. Since $U = \mathbb{P}^1 \setminus \Delta$ is an affine curve, we have $H^2(U, R^{2m-2}f_*\mathbb{Z}) = 0$. Hence

$$L^1H^{2m}(X_U) = \mathrm{Gr}^1_L H^{2m}(X_U) \cong H^1(U, H^{2m-1}_{\mathbb{Q}})$$

Recall that we have maps $\pi : \tilde{Y} \to Y$ and $r : \tilde{Y} \to X_U$. Define the *Griffiths homomorphism*

$$\mathrm{Griff} : H^{2m}_{\mathrm{pr}}(Y,\mathbb{Q}) \to H^1(U, H^{2m-1}_{\mathbb{Q}})$$

by $\mathrm{Griff}(\alpha) = (\pi \circ r)^*\alpha$.

Lemma 2.4 *If $Z \in Z^m(Y)$ and $[Z] \in H^{2m}_{\mathrm{pr}}(Y,\mathbb{Q})$ then $\mathrm{Griff}[Z]$ coincides with the cohomological invariant $\partial(\nu_Z)$ of the normal function ν_Z associated to Z.*

Proof: Set $Z_U = r^*\pi^*Z \in Z^m_{\mathrm{hom}}(X_U/U)$ and choose a covering of U by contractible open subsets U_i, $i \in I$. We can choose a family of cochains $\gamma^i(t)$ such that $\delta\gamma^i(t) = Z_U(t)$ for all $t \in U_i$. The 1–cocycle $\{\gamma^{ij}(t)\}$ defined by $\gamma^{ij}(t) = \gamma^i(t) - \gamma^j(t)$ represents the cycle class $\mathrm{cl}(Z_U) \in L^1H^{2m}(X_U,\mathbb{Q}) \cong H^1(U, H^{2m-1}_{\mathbb{Q}})$. We can perform a similar construction in homology: choose chains $\gamma_i(t)$ such that $\partial\gamma_i(t) = Z_i(t)$ and define $\gamma_{ij}(t) = \gamma_i(t) - \gamma_j(t)$. Let ω be a section of $(\mathcal{F}^m)^\vee$. We obtain a local lifting $\tilde{\nu}_i$ of ν_Z by setting

$$\langle \tilde{\nu}_i(t), \omega(t) \rangle = \int_{\gamma_i(t)} \omega(t)$$

The corresponding 1–cocycle $\{\tilde{\nu}_{ij}\} \in \check{C}^1(\mathcal{U}, (\mathcal{F}^m)^\vee)$ is given by integration along $\gamma_{ij}(t)$, hence it comes from $\{\gamma_{ij}\} \in \check{C}^1(U, H^{\mathbb{Z}}_{2m-1})$. Using Poincaré duality to identify the local systems $H^{\mathbb{Q}}_{2m-1}$ and $H^{2m-1}_{\mathbb{Q}}$ we find that $\partial(\nu_Z)$ is represented by the 1–cocycle $\{\gamma^{ij}\} \in \check{C}^1(U, H^{2m-1}_{\mathbb{Q}})$. □

Lemma 2.5 *The Griffiths homomorphism is injective if $H^{2m-1}_{\mathbb{Q}} \neq 0$.*

Proof: As U is an affine curve, the Leray spectral sequence induces an isomorphism

$$H^{2m}(X_U, \mathbb{Q}) \cong H^0(U, R^{2m} f_* \mathbb{Q}) \oplus H^1(U, H^{2m-1}_{\mathbb{Q}}).$$

As $[Z]$ maps to zero in the first summand, we have

$$\mathrm{Griff}[Z] = 0 \Longleftrightarrow \pi^*[Z] \in \ker(r^* : H^{2m}(\tilde{Y}) \to H^{2m}(X_U)).$$

Set $\Sigma = \bar{f}^{-1}(\Delta) \subset \tilde{Y}$. The exact sequence

$$H^{2m}_{\Sigma}(\tilde{Y}) \xrightarrow{\tau_*} H^{2m}(\tilde{Y}) \xrightarrow{r^*} H^{2m}(X_U)$$

shows that $\mathrm{Griff}[Z] = 0$ if and only if $\pi^*[Z] \in \mathrm{im}\, \tau_*$. By Poincaré–Lefschetz duality we have

$$H^{2m}_{\Sigma}(\tilde{Y}, \mathbb{Q}) \cong H_{2m}(\Sigma, \mathbb{Q}) \cong \oplus_{s \in \Delta} H_{2m}(X_s, \mathbb{Q}).$$

Let X_0 be a smooth fiber. By Picard–Lefschetz theory we have an exact sequence

$$0 \to H_{2m}(X_0) \to H_{2m}(X_s) \to \mathbb{Z} \xrightarrow{\partial} H_{2m-1}(X_0) \to H_{2m-1}(X_s) \to 0$$

and $\partial(1) = \delta_s$ is the vanishing cycle associated to the singular fiber X_s. As $H_{2m-1}(X_0)$ is generated by vanishing cycles, the hypothesis of the Lemma shows that there exists $s_0 \in \Delta$ such that $\delta_{s_0} \neq 0$, hence $H_{2m}(X_{s_0}) \cong H_{2m}(X_0)$. Since $H_{2m}(X_0) \cong H^{2m-2}(X_0) \cong H^{2m-2}(Y)$ by Poincaré duality and the Lefschetz hyperplane theorem, $H_{2m}(X_{s_0})$ can be identfied with $H^{2m-2}(Y)$. Under this identification $\pi_* \circ \tau_* : H^{2m-2}(Y) \to H^{2m}(Y)$ is identified with the Lefschetz operator L_Y, which is given by cup product with $c_1(\mathcal{O}_Y(1))$. As $\pi_* \pi^*[Z] = [Z]$ we find

$$\begin{aligned} \pi^*[Z] \in \mathrm{im}\, \tau_* &\Longleftrightarrow [Z] \in \mathrm{im}\, L_Y \\ &\Longleftrightarrow 0 = [Z] \in H^{2m}_{\mathrm{pr}}(Y). \end{aligned}$$

□

We can now finish the proof of Griffiths's theorem.

Proof: (Theorem 2.1) Suppose that $Z \in Z^m(Y)$ is a cycle such that $Z_t = Z \cap X_t$ is algebraically equivalent to zero for very general t. Then $\psi(Z_t) = 0$ for very general t by Lemma 2.3, hence the normal function ν_Z is zero. This implies that $\partial(\nu_Z) = 0$, hence $\mathrm{Griff}[Z] = 0$ by Lemma 2.4 and $[Z] = 0$ by Lemma 2.5. $\square$

We mention two applications of Theorem 2.1. Note that the theorem also applies to hypersurface sections (use a Veronese embedding to reduce to the case of hyperplane sections).

Corollary 2.6 *Let $Y \subset \mathbb{P}^{2m+1}$ ($m \geq 2$) be a smooth quadric and let X be the intersection of Y with a hypersurface of degree $d \geq 4$. If X is very general then* $\mathrm{Griff}^m(X) \otimes \mathbb{Q} \neq 0$.

Proof: The quadric Y contains two families of m–planes. Let L_1, L_2 be two m–planes from the two different families. If $d \geq 4$ the Hodge structure on $H^{2m-1}(X)$ is not of type $\{(m-1, m), (m, m-1)\}$ and we can apply Theorem 2.1 to the cycle $Z = L_1 - L_2$. (If $m \geq 3$ it suffices to take $d \geq 3$.) $\square$

Corollary 2.7 *Let Y be a smooth quintic fourfold such that* $\mathrm{im}(\mathrm{cl}^2_Y) \cap H^4_{\mathrm{pr}}(Y, \mathbb{Q}) \neq 0$ *and let $\{X_t\}_{t \in \mathbb{P}^1}$ be a Lefschetz pencil of hyperplane sections of Y. If t is very general then* $\mathrm{Griff}^2(X_t) \otimes \mathbb{Q} \neq 0$.

The Fermat quintic $Y \subset \mathbb{P}^5$ is an example of a quintic fourfold that satisfies the condition of the Corollary. It contains two planes P_1, P_2 such that $0 \neq [P_1 - P_2] \in H^4_{\mathrm{pr}}(Y, \mathbb{Q})$. By Corollary 2.7, the difference of the two lines $L_{1,t} = P_1 \cap H_t$ and $L_{2,t} = P_2 \cap H_t$ on the quintic threefold $X_t = Y \cap H_t$ is a nontorsion element of $\mathrm{Griff}^2(X_t)$ if t is very general. Note that the set of quintic fourfolds Y that satisfy the condition of Corollary 2.7 is a countable union of proper closed subsets of $\mathbb{P}H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}}(5))$ by the Noether–Lefschetz theorem.

Bibliographical hints. Griffiths's theorem appears in [13, Thm. 14.1]. A detailed study of the Griffiths homomorphism can be found in [18]. For the proof of Griffiths's theorem we have followed the arguments of Voisin [26].

3 The theorem of Green–Voisin

Let $S \subset \mathbb{P}^3$ be a surface of degree $d \geq 4$. If S is very general then $\operatorname{Pic}(S) \cong \mathbb{Z}$ by the Noether–Lefschetz theorem. Using the Lefschetz hyperplane theorem and the exponential sequence one easily proves that $\operatorname{Pic}(X) \cong \mathbb{Z}$ for every smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of dimension $n \geq 3$. Griffiths and Harris asked whether $\mathrm{CH}^2(X) \cong \mathbb{Z}$ if X is a very general threefold of degree $d \geq 6$ in $\mathbb{P}^4$. If this question has an affirmative answer, it would follow that the image of the Abel–Jacobi map ψ_X is zero if $d \geq 6$. Modulo torsion, this has been proved by Green and Voisin.

Theorem 3.1 (Green–Voisin) *Let $X \subset \mathbb{P}^{2m}$ be a smooth hypersurface of degree d and dimension $2m-1$ ($m \geq 2$). If X is very general and if $d \geq 2 + \frac{4}{m-1}$ then the image of*

$$\psi_X^m : \mathrm{CH}^m_{\mathrm{hom}}(X) \to J^m(X)$$

is contained in the torsion points of $J^m(X)$.

The proof of this result is obtained by a careful study of the infinitesimal invariant of a normal function. Again we need several lemmas, the first of which is a technique known as 'spreading out' an algebraic cycle.

Lemma 3.2 *Let $f : X \to S$ be a smooth projective morphism. If $Z_0 \in Z^m_{\mathrm{hom}}(X_{s_0})$ and $s_0 \in S$ is very general, then there exist a finite covering $g : T \to S$, a point $t_0 \in g^{-1}(s_0)$ and a relative cycle $Z_T \in Z^m_{\mathrm{hom}}(X_T/T)$ such that $Z_T(t_0) = Z_0$.*

Proof: There exists a relative Chow scheme $\mathrm{Chow}^m(X/S)$ that parametrises algebraic cycles of relative codimension m over S and a dominant map

$$p : \mathrm{Chow}^{\mathrm{m}}(X/S) \to S.$$

Let $\Sigma \subset S$ be the image of the irreducible components of $\mathrm{Chow}^m(X/S)$ that do not dominate S. The subset $\Sigma \subset S$ is a countable union of Zariski closed subsets of S. If $s_0 \in S \setminus \Sigma$ then there exists a finite morphism $g : T \to S$ such that $\mathrm{Chow}^m(X \times_S T/T)$ admits a rational section that passes through Z_0; let Z_T be its image. By construction the cycles $Z(t)$ are algebraically equivalent, hence homologically equivalent. □

Remark 3.3 By shrinking S and T, we may assume that $\mathrm{Chow}^m(X \times_S T/T)$ admits a section and that $g : T \to S$ is a finite étale morphism.

Set $V = H^0(\mathbb{P}^{2m}, \mathcal{O}_{\mathbb{P}}(d))$ and let $U \subset \mathbb{P}(V)$ be the complement of the discriminant locus. Consider the universal family of hypersurfaces

$$X_U = \{(x, F) \in \mathbb{P}^{2m} \times U | F(x) = 0\}.$$

Let $0 \in U$ be a base point and let $Z \in Z^m_{\mathrm{hom}}(X_0)$. If the base point is very general, we can apply Lemma 3.2 to the morphism $f : X_U \to U$ to 'spread out' Z_0 to a relative cycle $Z_T \in Z^m_{\mathrm{hom}}(X_T/T)$ over a new base T. After deleting the branch locus of the finite morphism $g : T \to U$ we may assume that g is étale. Let $\nu \in H^0(T, \mathcal{J}^m)$ be the normal function associated to Z_T.

Write $S = \mathbb{C}[X_0, \ldots, X_{2m}]$. Let $f \in S_d$ be a homogeneous polynomial of degree d, and let $J_f = (\frac{\partial f}{X_0}, \ldots, \frac{\partial f}{X_{2m}})$ be the Jacobian ideal of f. The quotient ring $R_f = S/J_f$ is a graded ring, called the Jacobi ring of f. Griffiths proved that the cohomology of $X = V(f)$ in the middle dimension can be described by the Jacobi ring:

$$H^{p,q}(X) \cong R_{(q+1)d-2m-1} \qquad (p + q = 2m - 1).$$

The following result is known as the *Symmetrizer Lemma*. We shall not prove it here, as we shall prove a more general result in the lectures on Nori's theorem.

Lemma 3.4 (Donagi–Green) *The Koszul complex*

$$\bigwedge^2 S_d \otimes S_{a-d} \to S_d \otimes S_a \to S_{a+d} \to 0$$

is exact if $a - d > 0$.

We shall use the symmetrizer lemma to study the cohomology sheaves of the complex

$$\Omega^\bullet(\mathcal{F}^m) = (\mathcal{F}^m \to \Omega^1_T \otimes \mathcal{F}^{m-1} \to \ldots).$$

Lemma 3.5 *If* $d \geq 2 + \frac{4}{m-1}$ *then* $\mathcal{H}^0(\Omega^\bullet(\mathcal{F}^m)) = \mathcal{H}^1(\Omega^\bullet(\mathcal{F}^m)) = 0$.

Proof: By a spectral sequence argument, it suffices to verify the assertion for the graded pieces

$$\mathrm{Gr}_F^p \Omega^\bullet(\mathcal{F}^m) = (\mathcal{H}^{p,2m-1-p} \to \Omega_T^1 \otimes \mathcal{H}^{p-1,2m-p} \to \ldots).$$

To this end, it suffices to show that the complex

$$0 \to H^{p,2m-p-1}(X_t) \to \Omega^1_{T,t} \otimes H^{p-1,2m-p}(X_t) \to \Omega^2_{T,t} \otimes H^{p-2,2m-p+1}(X_t)$$

is exact as far as written for all $p \geq m$ and for all $t \in T$. The dual complex is

$$\bigwedge^2 T_t \otimes H^{2m-p+1,p-2}(X_t) \to T_t \otimes H^{2m-p,p-1}(X_t) \to H^{2m-p-1,p}(X_t) \to 0.$$

As $g : T \to U$ is étale, we can identify the tangent space T_t with the tangent space to U at $g(t)$, which is isomorphic to S_d. Using Griffiths's description of the cohomology groups of X_t we can identify the dual complex with the complex

$$\bigwedge^2 S_d \otimes R_{(p-1)d-2m-1} \longrightarrow S_d \otimes R_{pd-2m-1} \longrightarrow R_{(p+1)d-2m-1}.$$

A diagram chase shows that this complex is exact at the middle term if

(i)
$$\bigwedge^2 S_d \otimes S_{(p-1)d-2m-1} \longrightarrow S_d \otimes S_{pd-2m-1} \longrightarrow S_{(p+1)d-2m-1}$$
is exact at the middle term;

(ii) the map $S_d \otimes J_{pd-2m-1} \longrightarrow J_{(p+1)d-2m-1}$ is surjective.

By the Symmetrizer Lemma, (i) holds for all $p \geq m$ if $(m-1)d \geq 2m+2$, which translates into the condition of the Lemma. As the Jacobian ideal J_f is generated in degree $d-1$, (ii) holds if $(m-1)d - 2m - 1 \geq d - 1$; this condition is weaker than the first condition. The multiplication map

$$S_d \otimes R_{pd-2m-1} \longrightarrow R_{(p+1)d-2m-1}$$

is surjective for all $p \geq m$ if $md \geq 2m+1$; again, this condition is weaker than the first condition. $\square$

The proof of Theorem 3.1 is finished by the following monodromy argument.

Lemma 3.6 *If $\mathcal{H}^0(\Omega^\bullet(\mathcal{F}^m)) = \mathcal{H}^1(\Omega^\bullet(\mathcal{F}^m)) = 0$ then ν is a torsion section of $\mathcal{J}^m$.*

Proof: As $\delta\nu$ is a global section of $\mathcal{H}^1(\Omega^\bullet(\mathcal{F}^m))$, we have $\delta\nu = 0$. Hence ν has flat local liftings. The vanishing of $\mathcal{H}^0(\Omega^\bullet(\mathcal{F}^m))$ implies that these flat local liftings are unique up to sections of the local system $H_\mathbb{Z} = R^{2m-1}f_*\mathbb{Z}/\mathrm{tors}$. Let

$$\rho : \pi_1(T, t_0) \to \mathrm{Aut}\, H^{2m-1}(X_0, \mathbb{C})$$

be the monodromy representation. By the uniqueness property of flat local liftings obtained above, it follows that

$$(*) \qquad \rho(\gamma)(\tilde{\nu}(0)) - \tilde{\nu}(0) \in H^{2m-1}(X_0, \mathbb{Z})$$

for all $\gamma \in \pi_1(T, t_0)$. To show that $\nu(0) \in J^m(X_0)$ is torsion, we have to prove that $\tilde{\nu}(0) \in H^{2m-1}(X_0, \mathbb{Q})$. We shall verify that this follows from $(*)$. Let $\{\gamma_i\}$ be a set of generators of $\pi_1(U, 0)$ coming from a Lefschetz pencil $L \subset U$, and let $\{\delta_i\}$ be the corresponding set of vanishing cocycles in $H^{2m-1}(X_0, \mathbb{Z})$. Let N be the index of the subgroup $g_*\pi_1(T, t_0) \subset \pi_1(U, 0)$. We have $\gamma_i^N = g_*\tilde{\gamma}_i$ some $\tilde{\gamma}_i \in \pi_1(T, t_0)$. By the Picard–Lefschetz formula we have

$$\rho(\tilde{\gamma}_i)(\tilde{\nu}(0)) - \tilde{\nu}(0) = \varepsilon N \langle \tilde{\nu}(0), \delta_i \rangle \delta_i$$

where $\varepsilon \in \{-1, 1\}$. Using $(*)$ we obtain $\langle \tilde{\nu}(0), \delta_i \rangle \in \mathbb{Q}$ for all i. As the vanishing cocycles generate $H^{2n-1}(X, \mathbb{Q})$, it follows that $\tilde{\nu}(0) \in H^{2m-1}(X_0, \mathbb{Q})$. $\square$

Remark 3.7 (i) Apart from the known exceptions $X = V(d) \subset \mathbb{P}^4$, $d \leq 5$, the only exceptions to the Green–Voisin theorem are cubic fivefolds and cubic sevenfolds. In both cases the image of the Abel–Jacobi map is nonzero modulo torsion; see [4] and [1].

(ii) Griffiths and Harris have shown that there are no nonzero normal functions over the complement $U \subset \mathbb{P}H^0(\mathbb{P}^4, \mathcal{O}_\mathbb{P}(d))$ of the discriminant locus if $d \geq 3$ (this is also true for $d = 2$, since the intermediate Jacobian of a quadric is zero). We have seen that there can be nonzero normal functions if $d \leq 5$, but they are multivalued sections of $\mathcal{J}^m$ that only become well–defined after passing to a finite covering $T \to U$.

Bibliographical hints. Theorem 3.1 was proved independently by M. Green and C. Voisin. Green's proof has appeared in [11]; see also [26]. The conjectures of Griffiths and Harris have appeared in [17]. The monodromy argument in Lemma 3.6 is taken from [27, Lecture 4].

4 Nori's connectivity theorem

Nori's connectivity theorem is a far–reaching generalisation of the theorem of Green–Voisin, based on the observation that their result can be interpreted in terms of the cohomology of the universal family of hypersurfaces in projective space.

Notation. Let $(Y, \mathcal{O}_Y(1))$ be a smooth polarised variety over $\mathbb{C}$ and let $X \subset Y$ be a smooth complete intersection of multidegree $(d_0, \ldots, d_r)$ and of dimension n. Set

$$E = \mathcal{O}_Y(d_0) \oplus \ldots \oplus \mathcal{O}_Y(d_r)$$

and define $S = \mathbb{P}H^0(Y, E)$. Let $U \subset S$ be the complement of the discriminant locus. Over S we have the universal family

$$X_S = \{(y, s) \in Y \times S | s(y) = 0\} \subset Y_S = Y \times S.$$

Given a morphism $T \to S$ we obtain induced familes

$$X_T = X \times_S T, \quad Y_T = Y \times T$$

over T by base change. Let $r : X_T \to Y_T$ be the inclusion map, and let $p : Y_T \to T$, $f = p \circ r : X_T \to T$ be the projection maps.

Nori's main result is the following connectivity theorem for the pair (Y_T, X_T).

Theorem 4.1 (Nori) *If* $\min(d_0, \ldots, d_r) \gg 0$ *then for every smooth morphism* $T \to S$ *we have* $H^{n+k}(Y_T, X_T, \mathbb{Q}) = 0$ *for all* $k \leq n$.

Remark 4.2 (i) For every base change $T \to S$ (not necessarily smooth) we have $H^k(Y_T, X_T, \mathbb{Z}) = 0$ for all $k \leq n$. To see this, note that the Lefschetz hyperplane theorem shows that the restriction map $i^* R^q p_* \mathbb{Z} \to$

$R^q f_*\mathbb{Z}$ is an isomorphism if $q < n$ and is injective if $q = n$. As an exercise, the reader may verify that this implies that the restriction map $H^k(Y_T, \mathbb{Z}) \to H^k(X_T, \mathbb{Z})$ is an isomorphism if $k \leq n-1$ and is injective if $k = n$ by comparing the Leray spectral sequence

$$E_2^{p,q} = H^p(T, R^q f_*\mathbb{Z}) \Rightarrow H^{p+q}(X_T, \mathbb{Z})$$

to the Künneth spectral sequence

$$\tilde{E}_2^{p,q} = H^p(T, \mathbb{Z}) \otimes H^q(Y, \mathbb{Z}) \Rightarrow H^{p+q}(Y_T, \mathbb{Z}).$$

If $k \geq 1$ one can construct examples where $H^{n+k}(Y_T, X_T, \mathbb{Z})$ is nonzero even if $\min(d_0, \ldots, d_r) \gg 0$.

(ii) Set $\mathcal{E} = p_Y^* E \otimes p_S^* \mathcal{O}_S(1)$. The variety $X_S \subset Y_S$ is the zero locus of the tautological section $\tau \in H^0(Y_S, \mathcal{E})$. As $\mathcal{E}$ is an ample vector bundle on the projective variety Y_S, it follows that $H^k(Y_S, X_S, \mathbb{Z}) = 0$ for all $p \leq \dim X_S$ [19]. This connectivity result is stronger than the statement in Nori's theorem for $T = S$, but it is not invariant under base change.

(iii) The local system $R^n f_*\mathbb{Q}$ splits as a direct sum of a fixed part $i^* R^n p_*\mathbb{Q}$ and a variable part $\mathbb{V}$. Nori's connectivity theorem is equivalent to a statement about cohomology with values in the local system $\mathbb{V}$. This follows from Deligne's theorem on the degeneration at E_2 of the Leray spectral sequence for f with $\mathbb{Q}$–coefficients. Define

$$H^k(Y)_0 = \ker(i^* : H^k(Y) \to H^k(X)).$$

Note that this group is zero if $k \leq n$. By Deligne's theorem, we have

$$H^{n+k}(X_T, \mathbb{Q}) \cong \bigoplus\nolimits_{p+q=n+k} H^p(T, R^q f_*\mathbb{Q}).$$

Comparing the decomposition of $H^{n+k}(X_T)$ with the Künneth decomposition of $H^{n+k}(Y_T)$ we find that $H^{n+k}(Y_T, X_T) = 0$ for all $k \leq c$ if and only if the map

$$\bigoplus\nolimits_{i=0}^{k-1} H^i(T, \mathbb{Q}) \otimes H^{n+k-i}(Y, \mathbb{Q})_0 \to H^k(T, \mathbb{V})$$

is an isomorphism for all $k < c$ and is injective for $k = c$.

(iv) Nori's theorem does not necessarily hold if we omit the smoothness assumption on the base change. For instance, let $L \subset S$ be a Lefschetz pencil of hypersurface sections on a smooth projective variety Y such that $\dim Y = 2m$ and $H^{2m}_{\mathrm{pr}}(Y, \mathbb{Q}) \neq 0$. Let $T = U \cap L$ be the smooth part of the Lefschetz pencil. Using (iii), the vanishing of $H^{2m}(Y_T, X_T)$ is equivalent to the vanishing of $H^0(T, \mathbb{V})$ and the injectivity of the map

$$\psi_1 : H^{2m}_{\mathrm{pr}}(Y, \mathbb{Q}) \to H^1(T, \mathbb{V}).$$

By Picard–Lefschetz theory we have $H^0(T, \mathbb{V}) = 0$. As the map ψ_1 coincides with the Griffiths homomorphism discussed in Lecture 2, it follows from Griffiths's theorem that $H^{2m}(Y_T, X_T) = 0$ if $d \gg 0$. Again using (iii), the vanishing of $H^{2m+1}(Y_T, X_T, \mathbb{Q})$ would imply that there is an injective map

$$\psi_2 : H^{2m+1}(Y, \mathbb{Q})_0 \oplus H^1(T, \mathbb{Q}) \otimes H^{2m}_{\mathrm{pr}}(Y, \mathbb{Q}) \to H^2(T, \mathbb{V}).$$

As T is affine, we have $H^2(T, \mathbb{V}) = 0$. As $H^1(T, \mathbb{Q})$ is nonzero if $d \gg 0$, the left hand side is nonzero for $d \gg 0$; hence ψ_2 cannot be injective and $H^{2m+1}(Y_T, X_T, \mathbb{Q}) \neq 0$ for all $d \gg 0$. A similar argument shows that $H^{n+c}(Y_T, X_T) \neq 0$ if $L \subset S$ is a general linear subspace of dimension $e < c$. By analogy with Griffiths's theorem, Nori [24, Conj. 7.4.1] conjectures that $H^{n+k}(Y_T, X_T) = 0$ for all $k \leq c$ if L is a general linear subspace of dimension c in S.

The following result shows that Nori's theorem implies the theorem of Green–Voisin.

Theorem 4.3 *Let X be a very general smooth complete intersection in Y of multidegree $(d_0, \dots, d_r)$ and dimension n, with inclusion map $i : X \to Y$. If $\min(d_0, \dots, d_r) \gg 0$ then*

$$\mathrm{im}(\mathrm{cl}^p_{\mathcal{D},X}) \subset i^* H^{2p}_{\mathcal{D}}(Y, \mathbb{Q}(p))$$

for all $p < n$.

Proof: Let $\Delta \subset S$ be the discriminant locus. If $s_0 \in S$ is very general there exist for every $z_0 \in \mathrm{CH}^p_{\mathrm{hom}}(X_{s_0})$ a subset $U' \subset S \setminus \Delta$ containing s_0, a finite étale covering $g : T \to U'$, a relative cycle $Z \in \mathrm{CH}^p_{\mathrm{hom}}(X_T/T)$ and $t_0 \in g^{-1}(s_0)$ such that $Z_T(t_0) = z_0$. Consider the commutative diagram

$$\begin{array}{ccc} H^{2p}_{\mathcal{D}}(Y_T, \mathbb{Q}(p)) & \xrightarrow{r^*} & H^{2p}_{\mathcal{D}}(X_T, \mathbb{Q}(p)) \\ \Big\downarrow k^* & & \Big\downarrow j^* \\ H^{2p}_{\mathcal{D}}(Y, \mathbb{Q}(p)) & \xrightarrow{i^*} & H^{2p}_{\mathcal{D}}(X_{s_0}, \mathbb{Q}(p)). \end{array}$$

It follows from Theorem 4.1 and the long exact sequence of Deligne cohomology (Appendix A) that

$$H^{n+k}_{\mathcal{D}}(Y_T, X_T, \mathbb{Q}(p)) = 0$$

for all $k \leq n$. Hence the map r^* is surjective for all $p \leq n-1$. Choose $\xi \in H^{2p}_{\mathcal{D}}(Y_T, \mathbb{Q}(p))$ such that $r^*\xi = \mathrm{cl}_{\mathcal{D}}(Z_T)$ and put $\eta = k^*\xi \in H^{2p}_{\mathcal{D}}(Y, \mathbb{Q}(p))$. By construction we have $i^*\eta = j^*\,\mathrm{cl}_{\mathcal{D}}(Z_T) = \mathrm{cl}_{\mathcal{D}}(z_0)$. □

Remark 4.4 (i) By the Lefschetz hyperplane theorem, the statement of the theorem is only nontrivial if $n = 2p$ or $n = 2p-1$. Let us take $Y = \mathbb{P}^N$. If we apply the theorem with $n = 2p$, we find that the image of cl^p_X is isomorphic to $\mathbb{Z}$ (note that $H^{2p}(X, \mathbb{Z})$ is torsion free), which is the general form of the Noether–Lefschetz theorem. If we apply the theorem with $n = 2p-1$, we find that the image of the Abel–Jacobi map ψ^p_X is contained in the torsion points of $J^p(X)$, which is the Green–Voisin theorem. Note that we only obtain asymptotic versions of these theorems. Paranjape [25] has obtained an effective version of Nori's theorem. It leads to the bound $d \geq 2p+2$ in both cases. This bound is optimal if $n = 2p$ and $p = 1$ and if $n = 2p-1$ and $p = 2$, but not in general: we obtained the better bound $d \geq 2 + \frac{4}{p-1}$ for the Green–Voisin theorem in Lecture 3. We shall see later how to obtain effective degree bounds for Theorem 4.1 that do give the optimal bounds for the theorems of Noether–Lefschetz and Green–Voisin.

(ii) Theorem 4.3 shows that we cannot expect to obtain a connectivity result for the pair (Y_T, X_T) without conditions on the degrees. For instance we cannot have $H^5(\mathbb{P}^4_T, X_T) = 0$ if $d \leq 5$ since the image of the Abel–Jacobi map on a very general quintic hypersurface $X \subset \mathbb{P}^4$ is not contained in the torsion points of $J^2(X)$ by Griffiths's theorem.

(iii) Theorem 4.3 does not hold for zero cycles ($p = n$). Consider for example a family $C_T \to T$ of smooth plane curves of degree d ($n = 1$). If Theorem 4.3 could be applied to this case, we would find that the image of the Abel–Jacobi map for a very general plane curve is contained in the torsion points of the Jacobian. This is clearly false by the Jacobi inversion theorem.

Bibliographical hints. Nori's theorem has appeared in [24]; see also [12, Lecture 8]. Paranjape's effective version of Nori's theorem can be found in [25]. Green and Müller–Stach have obtained a more precise version of Theorem 4.3; see [16].

5 Sketch of proof of Nori's theorem

In this section we sketch the proof of an effective version of Nori's theorem along the lines of the proof of Green–Voisin. Our condition on the base change is more restrictive than the one in Nori's original result: we consider smooth morphisms $T \to S$ such that the induced map $X_T \to T$ is smooth, i.e., we demand that the morphism $T \to S$ factors through the complement of the discriminant locus. This suffices for the geometric applications that we shall discuss in the next lecture.

We start by recalling a little bit of mixed Hodge theory. The cohomology groups of a quasi–projective variety do not always carry a pure Hodge structure (HS). Consider for example the variety $U = \mathbb{P}^1(\mathbb{C}) \setminus \{0, \infty\}$. As U is homotopically equivalent to a circle its first Betti number is 1, hence $H^1(U)$ cannot carry a pure HS. (A similar result holds for the complement of two points in a smooth compact curve of arbitrary genus.) Deligne has proved that the cohomology groups of smooth quasi–projective variety X always carry a *mixed Hodge structure* (MHS). This means that for every $k \geq 0$ there exist a decreasing filtration $F^\bullet$ on $H^k(X, \mathbb{C})$ (the Hodge filtration) and an increasing filtration $W_\bullet$ on $H^k(X, \mathbb{Q})$ (the weight filtration) such that $F^\bullet$ induces a pure HS of weight m on the graded pieces

$$\mathrm{Gr}_m^W H^k(X) = W_m H^k(X)/W_{m-1} H^k(X)$$

of the weight filtration. By construction, the weight filtration on the cohomology of a smooth quasi–projective variety satisfies

$$\mathrm{Gr}_m^W H^k(X) = 0 \quad \text{if } m < k.$$

A morphism of MHS $f : H_1 \to H_2$ is a homomorphism of abelian groups that is compatible with the filtrations $F^\bullet$ and $W_\bullet$. Deligne has shown that such morphisms are *strictly compatible* with the Hodge and weight filtrations, i.e.

$$F^p H_2 \cap \operatorname{im} f = f(F^p H_1)$$
$$W_m H_2 \cap \operatorname{im} f = f(W_m H_1).$$

The proof of Nori's theorem proceeds in several steps.

Step 1: mixed Hodge theory. As we usually want to apply Nori's theorem to a quasi–projective base T, we have to deal with the cohomology of the quasi–projective varieties X_T and Y_T. By Deligne's theorem the groups $H^{n+k}(Y_T)$ and $H^{n+k}(X_T)$ carry a MHS. Using a cone construction one can also put a MHS on the relative cohomology $H^{n+k}(Y_T, X_T)$. As the long exact sequence

$$H^{n+k-1}(X_T) \to H^{n+k}(Y_T, X_T) \to H^{n+k}(X_T) \to H^{n+k}(X_T)$$

is an exact sequence of MHS and morphisms of MHS are strictly compatible with the weight filtration, we have

$$\operatorname{Gr}_m^W H^{n+k}(Y_T, X_T) = 0 \quad \text{if } m < n + k - 1.$$

The presence of a MHS on $H^{n+k}(Y_T, X_T)$ implies that this vector space vanishes if a large enough subspace of it vanishes.

Lemma 5.1 *Suppose there exists a natural number $m \le [\frac{n+k}{2}]$ such that $F^m H^{n+k}(Y_T, X_T) = 0$. Then $H^{n+k}(Y_T, X_T) = 0$.*

Proof: Suppose that $H^{n+k}(Y_T, X_T) \neq 0$. Then there exists $i \ge n + k - 1$ such that $\operatorname{Gr}_i^W H^{n+k}(Y_T, X_T) \neq 0$. We have a Hodge decomposition

$$\operatorname{Gr}_i^W H^{n+k}(Y_T, X_T) = \oplus_{p+q=i} H^{p,q}$$

such that $H^{q,p} = \bar{H^{p,q}}$ (Hodge symmetry). If $H^{p,q} \neq 0$ then $p \le m - 1$ by the hypothesis of the Lemma, hence also $q \le m - 1$ by Hodge symmetry. But then $i = p + q \le 2m - 2 \le n + k - 2$, contradiction. $\square$

It is difficult to apply the previous Lemma to prove the vanishing of $H^{n+k}(Y_T, X_T)$ since the Hodge filtration on $H^{n+k}(Y_T, X_T)$ is defined in a complicated way. One starts by choosing compatible *good compactifications* $\bar{Y}_T$ and $\bar{X}_T$ of Y_T and X_T with boundary divisors $\tilde{D}_T = \bar{Y}_T \backslash Y_T$, $D_T = \bar{X}_T \backslash X_T$. Let $j : \bar{X}_T \to \bar{Y}_T$ be the inclusion map (its restriction to X_T is also denoted by j) and let $C^\bullet(\alpha)$ be the cone of the (surjective) map

$$\alpha : \Omega^\bullet_{\bar{Y}_T}(\log \tilde{D}_T) \to j_* \Omega^\bullet_{\bar{X}_T}(\log D_T).$$

One can show that $\mathbb{H}^{n+k}(C^\bullet(\alpha)) \cong H^{n+k}(Y_T, X_T)$. The Hodge filtration on $H^{n+k}(Y_T, X_T)$ is defined by

$$F^p H^{n+k}(Y_T, X_T) = \operatorname{im}(\mathbb{H}^{n+k}(\sigma_{\geq p} C^\bullet(\alpha)) \to \mathbb{H}^{n+k}(C^\bullet(\alpha))).$$

There is another, easier way to put a filtration on $H^{n+k}(Y_T, X_T)$. Define

$$\Omega^\bullet_{Y_T, X_T} = \ker(\beta : \Omega^\bullet_{Y_T} \to j_* \Omega^\bullet_{X_T}).$$

It follows from Grothendieck's algebraic de Rham theorem and the five lemma that $\mathbb{H}^{n+k}(\Omega^\bullet_{Y_T, X_T}) \cong H^{n+k}(Y_T, X_T)$. Hence we can define a second filtration $G^\bullet$ on $H^{n+k}(Y_T, X_T)$ by

$$G^p H^{n+k}(Y_T, X_T) = \operatorname{im}(\mathbb{H}^{n+k}(\sigma_{\geq p} \Omega^\bullet_{Y_T, X_T}) \to \mathbb{H}^{n+k}(\Omega^\bullet_{Y_T, X_T})).$$

As β is surjective, the complex $\Omega^\bullet_{Y_T, X_T}$ is quasi–isomorphic to $C^\bullet(\beta)$. The restriction from $\bar{Y}_T$ to Y_T induces a quasi–isomorphism $C^\bullet(\alpha) \to C^\bullet(\beta)$. Hence

$$F^p H^{n+k}(Y_T, X_T) \subseteq G^p H^{n+k}(Y_T, X_T),$$

so it suffices to show that $G^m H^{n+k}(Y_T, X_T) = 0$ for some $m \leq [\frac{n+k}{2}]$. The advantage of working with $G^\bullet$ is that we can work on Y_T and X_T and do not have to pass to a compactification.

Step 2: semicontinuity. There exists a spectral sequence (induced by the filtration bête)

$$E_1^{a,b} = H^a(Y_T, \Omega^b_{Y_T, X_T}) \Rightarrow \mathbb{H}^{a+b}(\Omega^\bullet_{Y_T, X_T}).$$

Using this spectral sequence we find that $G^m H^{n+k}(Y_T, X_T) = 0$ if

$$H^a(Y_T, \Omega^b_{Y_T, X_T}) = 0 \ \text{ for all } (a, b) \text{ such that } a + b \leq n + k, \ b \geq m.$$

Let $p : Y_T \to T$ be the projection map. By the Leray spectral sequence it suffices to show that $H^i(T, R^j p_* \Omega^b_{Y_T,X_T}) = 0$ for all (i,j) such that $i + j = a$. This is certainly true if

$$R^j p_* \Omega^b_{Y_T,X_T} = 0$$

for all $j \leq a$. Let $i_t : Y \to Y_T$ be the inclusion map defined by $i_t(y) = (y,t)$. As the maps $p : Y_T \to T$ and $f : X_T \to T$ are flat and the sheaves $\Omega^b_{Y_T}$ and $\Omega^b_{X_T}$ are locally free, it follows that these sheaves are flat over $\mathcal{O}_T$. Hence $\Omega^b_{Y_T,X_T}$ is flat over $\mathcal{O}_T$. By semicontinuity, $R^j p_* \Omega^b_{Y_T,X_T} = 0$ if

$$H^j(Y, i_t^* \Omega^b_{Y_T,X_T}) = 0$$

for all $t \in T$.

Step 3: Leray filtration. Suppose that $f : X_T \to T$ is smooth. In this case we have an exact sequence

$$0 \to f^* \Omega^1_T \to \Omega^1_{X_T} \to \Omega^1_{X_T/T} \to 0.$$

The *Leray filtration* $L^\bullet$ on $\Omega^b_{X_T}$ is defined by

$$L^p \Omega^b_{X_T} = \operatorname{im}(f^* \Omega^p_T \otimes \Omega^{b-p}_{X_T} \to \Omega^b_{X_T}).$$

Its graded pieces are

$$\operatorname{Gr}^p_L \Omega^b_{X_T} \cong f^* \Omega^p_T \otimes \Omega^{b-p}_{X_T/T}.$$

The spectral sequence associated to the induced filtration on $\Omega^b_{X_T} \otimes \mathcal{O}_{X_t}$ is

$$E_1^{p,q} = \Omega^p_{T,t} \otimes H^{p+q}(X_t, \Omega^{b-p}_{X_t}) \Rightarrow H^{p+q}(X_t, \Omega^b_{X_T} \otimes \mathcal{O}_{X_t}).$$

One can show that the d_1 map in this spectral sequence is the differential of the period map; it is given by cup product with the Kodaira–Spencer class. The Leray filtration on $\Omega^\bullet_{Y_T}$ splits, as Y_T is a product. Define a filtration $L^\bullet$ on $\Omega^\bullet_{Y_T,X_T}$ by

$$L^p \Omega^b_{Y_T,X_T} = \ker(L^p \Omega^b_{Y_T} \to L^p j_* \Omega^b_{X_T}).$$

Set $\Omega^b_{(Y_T,X_T)/T} = \ker(\Omega^b_{Y_T/T} \to j_* \Omega^b_{X_T/T})$. We have

$$\operatorname{Gr}^p_L \Omega^b_{Y_T,X_T} \cong f^* \Omega^p_T \otimes \Omega^{b-p}_{(Y_T,X_T)/T}.$$

If we restrict $L^\bullet$ to the fiber $Y \times \{t\}$ we obtain a spectral sequence

$$E_1^{p,q}(b) = \Omega^p_{T,t} \otimes H^{p+q}(Y, \Omega^{b-p}_{Y,X_t}) \Rightarrow H^{p+q}(Y, i_t^* \Omega^b_{Y_T,X_T}).$$

Using the semicontinuity result from Step 2, we find that $H^{n+k}(Y_T, X_T) = 0$ if

$$E_\infty^{p,q}(b) = 0 \text{ for all } (p,q,b) \text{ such that } p+q+b \leq n+k,\ b \geq m.$$

Step 4: reduction to hypersurfaces. There exists a trick using projective bundles to reduce questions about complete intersections to hypersurfaces. It was used by Terasoma and Konno to define Jacobi rings for complete intersections in projective space. We can apply a similar trick to the relative cohomology of the pair (Y_T, X_T). Set

$$\mathcal{E} = p_Y^* E \otimes p_T^* \mathcal{O}_T(1)$$

and let $P_T = \mathbb{P}(\mathcal{E}^\vee)$ be the projective bundle associated to $\mathcal{E}^\vee$ with projection map $\pi_T : P_T \to Y_T$. On P_T there exists a tautological line bundle ξ such that $H^0(P_T, \xi) \cong H^0(Y_T, \mathcal{E})$. We know that $X_T \subset Y_T$ is the zero locus of a section $\sigma \in H^0(Y_T, \mathcal{E})$. Let $\tilde{\sigma}$ be the corresponding section of ξ, and let $\tilde{X}_T \subset P_T$ be its zero locus.

Lemma 5.2 *For all $k \geq 0$ there is an isomorphism*

$$H^k(Y_T, X_T) \cong H^{k+2r}(P_T, \tilde{X}_T).$$

Proof: Consider the diagram

$$\begin{array}{ccccccc}
\pi_T^{-1}(X_T) = \mathbb{P}(\mathcal{E}^\vee|_{X_T}) & \subset & \tilde{X}_T & \subset & P_T & \supset & P_T \setminus \tilde{X}_T \\
 & & & & \downarrow \pi_T & & \downarrow \pi_T \\
 & & X_T & \subset & Y_T & \supset & Y_T \setminus X_T.
\end{array}$$

As the line bundle ξ restricts to $\mathcal{O}_{\mathbb{P}}(1)$ on each fiber of π_T, the induced map

$$\pi_T : P_T \setminus \tilde{X}_T \to Y_T \setminus X_T$$

is a fiber bundle with fiber $\mathbb{A}^r$. Hence $(\pi_T)_*$ induces an isomorphism

$$H_c^{k+2r}(P_T \setminus \tilde{X}_T) \cong H_c^k(Y_T \setminus X_T).$$

By Poincaré–Lefschetz duality we find an isomorphism $H^{k+2r}(P_T, \tilde{X}_T) \cong H^k(Y_T, X_T)$. $\square$

By Lemma 5.2 it suffices to prove Nori's theorem for a family of hypersurface sections $X_T \subset Y_T$ defined by sections of a very ample line bundle $L = \mathcal{O}_Y(d)$.

Step 5: base change. We say that Nori's condition (N_c) holds if

$$R^a p_* \Omega^b_{Y_T, X_T} = 0 \ \text{ for all } (a, b) \text{ such that } a + b \leq n + c, \ b \geq m.$$

If (N_c) holds, then $H^{n+k}(Y_T, X_T) = 0$ for all $k \leq c$.

Lemma 5.3 (Nori) *Let U and T be smooth quasi–projective varieties and let $g : T \to U$ be a smooth morphism.*

(i) If (N_c) holds for U, then (N_c) holds for T;

(ii) if g is smooth and surjective and (N_c) holds for T, then (N_c) holds for U.

For the proof of Lemma 5.3, see [24, Lemma 2.2].

Define $V = H^0(Y, L)$, $S = \mathbb{P}(V)$. Let $\Delta \subset S$ and $\Delta' \subset V$ be the discriminant loci and let $U = S \setminus \Delta$, $U' = V \setminus \Delta'$ be their complements. Let $i : U' \to V \setminus \{0\}$ be the inclusion, and let $\pi : V \setminus \{0\} \to \mathbb{P}(V)$ be the projection. The composed map $\pi \circ i : U' \to S$ is a smooth morhpism with image U. If (N_c) holds for the base U', then (N_c) holds for every base T such that $T \to U$ is a smooth morphism by Lemma 5.3. Hence it suffices to prove that (N_c) holds for one particular choice of the base T, namely $T = U'$. In this case, the tangent space T_t to T at every point t can be identified with V. Hence

$$E_1^{p,q}(b) \cong \bigwedge\nolimits^p V^\vee \otimes H^{p+q}(\Omega^{b-p}_{Y, X_t})$$

for all $t \in T$. There exists a perfect pairing

$$\Omega^p_{Y, X_t} \otimes \Omega^{n+1-p}_Y(\log X_t) \to K_Y$$

given by wedge product. Using this pairing we can identify $K_Y \otimes (\Omega^p_{Y, X_t})^\vee$ with $\Omega^{n+1-p}_Y(\log X_t)$. By Serre duality the dual of $E_1^{p,q}(b)$ is

$$E_1^{-p, n+1-q}(b) = \bigwedge\nolimits^p V \otimes H^{n+1-p-q}(Y, \Omega^{n+1-b+p}_Y(\log X_t)).$$

Step 6: Green's generalised Jacobi ring. Let $P^1(L)$ be the first jet bundle of L. It fits into an exact sequence

$$0 \to \Omega^1_Y \otimes L \to P^1(L) \to L \to 0. \tag{2}$$

There exists a map $j^1 : L \to P^1(L)$ that associates to a section s of L its 1–jet $j^1(s)$. If we dualise the exact sequence (2) and tensor it by L we obtain an exact sequence

$$0 \to \mathcal{O}_Y \to \Sigma_{Y,L} \to T_Y \to 0 \tag{3}$$

with extension class $2\pi i c_1(L) \in H^1(Y, \Omega^1_Y)$. The bundle $\Sigma_{Y,L}$ is called the first prolongation bundle of L (bundle of first order differential operators on sections of L).

Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_Y$–modules. For every $s \in H^0(Y, L)$ we have a map $\Sigma_{Y,L} \otimes L^{-1} \to \mathcal{F}$ that is given by contraction with $j^1(s) \in P^1(L)$. Let

$$g_s : H^0(Y, \Sigma_{Y,L} \otimes L^{-1}) \to H^0(Y, \mathcal{F})$$

be the induced map on global sections. Define

$$J_{Y,s}(\mathcal{F}) = \operatorname{im} g_s, \quad R_{Y,s}(\mathcal{F}) = \operatorname{coker} g_s.$$

Let $X \in |L|$ be a smooth hypersurface section of Y of dimension n. The Poincaré residue sequence

$$0 \to \Omega_Y^{n-p+1} \to \Omega_Y^{n-p+1}(\log X) \xrightarrow{\text{Res}} i_*\Omega_X^{n-p} \to 0$$

induces an exact sequence

$$0 \to H_{\text{pr}}^{n-p+1,p}(Y) \to H^p(Y, \Omega_Y^{n+1-p}(\log X)) \to H_{\text{var}}^{n-p,p}(X) \to 0.$$

Recall that a property (P) is said to hold for a *sufficiently ample* line bundle L if there exists a line bundle L_0 such that (P) holds for L if $L \otimes L_0^{-1}$ is ample.

Proposition 5.4 (Green) *If L is sufficiently ample then*

$$H^p(Y, \Omega_Y^{n-p+1}(\log X)) \cong R_{Y,s}(K_Y \otimes L^{p+1}).$$

Proof: Contraction with the 1–jet $j^1(s)$ defines a map $\Sigma_L \to L$ whose kernel is isomorphic to $T_Y(-\log X)$. If we dualise and take exterior powers in the resulting short exact sequence

$$0 \to T_Y(-\log X) \to \Sigma_L \to L \to 0,$$

we obtain a long exact seqeunce

$$\begin{gathered} 0 \to \Omega_Y^{n+1-p}(\log X) \to \bigwedge^{n-p+2}\Sigma_L^\vee \otimes L \to \dots \\ \dots \to \bigwedge^{n+1}\Sigma_L^\vee \otimes L^p \to \bigwedge^{n+2}\Sigma_L^\vee \otimes L^{p+1} \to 0. \end{gathered}$$

Using the identifications $\bigwedge^{n+1}\Sigma_L^\vee \cong K_Y \otimes \Sigma_L$ and $\bigwedge^{n+2}\Sigma_L^\vee \cong K_Y$, we obtain the isomorphism of the Proposition by chasing through the spectral sequence of hypercohomology associated to this long exact sequence. □

The bigraded ring

$$R_{Y,s} = \oplus_{p,q\geq 0} R_{Y,s}(Y, K_Y^p \otimes L^{q+1})$$

is called the *Jacobi ring* associated to s.

Example 5.5 Take $Y = \mathbb{P}^{n+1}$, $L = \mathcal{O}_{\mathbb{P}}(d)$. The exact sequence (3) is the familiar Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}} \to \oplus^{n+2}\mathcal{O}_{\mathbb{P}}(1) \to T_{\mathbb{P}} \to 0.$$

Using this sequence, one checks that $R_{Y,s}(K_Y \otimes L^{p+1}) \cong R_{(p+1)d-n-2}$. Hence the ring $\oplus_p R_{Y,s}(K_Y \otimes L^{p+1})$ coincides with Griffiths's Jacobi ring

Step 7: Koszul cohomology. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_Y$–modules. The Koszul cohomology group $\mathcal{K}_{p,q}(\mathcal{F}, L)$ is the cohomology group at the middle term of the complex

$$\bigwedge^{p+1}V \otimes H^0(\mathcal{F} \otimes L^{q-1}) \to \bigwedge^p V \otimes H^0(\mathcal{F} \otimes L^q) \to \bigwedge^{p-1}V \otimes H^0(\mathcal{F} \otimes L^{q+1}).$$

These groups were introduced and studied by M. Green.

A standard technique to obtain vanishing theorems for Koszul cohomology is due to Green and Lazarsfeld. Let M_L be the kernel of the surjective evaluation map $e_L : V \otimes \mathcal{O}_Y \to L$. It fits into an exact sequence

$$0 \to M_L \to V \otimes_{\mathbb{C}} \mathcal{O}_Y \to L \to 0.$$

If we take exteriors powers in this short exact sequence and twist by $\mathcal{F}\otimes L^{q-1}$ we obtain a complex

$$0 \to \bigwedge^{p+1} M_L \otimes \mathcal{F} \otimes L^{q-1} \to \bigwedge^{p+1} V \otimes \mathcal{F} \otimes L^{q-1} \to \bigwedge^{p} V \otimes \mathcal{F} \otimes L^{q} \to \\ \to \bigwedge^{p-1} V \otimes \mathcal{F} \otimes L^{q+1} \to \ldots \to \mathcal{F} \otimes L^{p+q} \to 0.$$

The Koszul complex is obtained from this complex by taking global sections, and one obtains the following result.

Proposition 5.6

$$\mathcal{K}_{p,q}(\mathcal{F}, L) = 0 \textit{ if } H^1(Y, \bigwedge^{p+1} M_L \otimes \mathcal{F} \otimes L^{q-1}) = 0.$$

Step 8: the double complex. By Proposition 5.4 we can identify

$$E_1^{p,q}(b)^\vee = E_1^{-p,n+1-q}(b) = \bigwedge^p V \otimes H^{n+1-p-q}(Y, \Omega_Y^{n+1-b+p}(\log X_t))$$

with $\bigwedge^p V \otimes R_{Y,t}(K_Y \otimes L^{b-p+1})$. We can identify this E_1 term with the E_1 term of another spectral sequence. Consider the double complex $\mathcal{B}^{\bullet,\bullet}(b)$ defined by

$$\mathcal{B}^{-i,j}(b) = \bigwedge^i V \otimes K_Y \otimes \bigwedge^{b-j} \Sigma_{Y,L} \otimes L^{j-i+1}, \quad j-i \geq 0.$$

The complex $\mathcal{B}^{\bullet,\bullet}(b)$ is a second quadrant double complex which consists of the terms $\mathcal{B}^{-i,j}(b)$ with $0 \leq i \leq b$, $0 \leq j \leq b$ and $j - i \geq 0$:

$$\begin{array}{ccccc}
\bigwedge^b V \otimes K_Y \otimes L & \to & \ldots & \to & K_Y \otimes L^{b+1} \\
 & & & & \uparrow \\
 & \ddots & & & \vdots \\
 & & \ddots & & \uparrow \\
 & & & & \bigwedge^{n+2-b} \Sigma_{Y,L}^\vee \otimes L.
\end{array}$$

The horizontal differential in this complex is the differential of the Koszul complex; the vertical differential is given by contraction with the 1–jet $j^1(s)$. Let $\mathcal{B}^\bullet(b) = s(\mathcal{B}^{\bullet,\bullet}(b))$ be the associated total complex. Set

$$B^{-i,j}(b) = H^0(Y, \mathcal{B}^{-i,j}(b)), \quad B^k(b) = H^0(Y, \mathcal{B}^k(b)).$$

We have two spectral sequences associated to $B^{\bullet,\bullet}(b)$, given by filtering along the rows or columns:

$$'E_1^{p,q}(b) = H^q(B^{p,\bullet}(b)) \Rightarrow H^{p+q}(B^\bullet(b))$$
$$''E_1^{p,q}(b) = H^q(B^{\bullet,p}(b)) \Rightarrow H^{p+q}(B^\bullet(b)).$$

By definition

$$'E_1^{-p,n+1-q}(b) = \bigwedge^p V \otimes R_{Y,t}(K_Y \otimes L^{b-p+1}).$$

It is not difficult to show that the isomorphism

$$E_1^{-p,n+1-q}(b) \cong {}'E_1^{-p,n+1-q}(b)$$

is compatible with the d_1 maps in both spectral sequences; hence it induces an isomorphism on the E_2 terms. But this does not suffice to get an isomorphism on the E_∞ terms. In [23] we constructed a morphism of filtered complexes $\mathcal{B}^\bullet(b) \to \mathcal{C}^\bullet(b)$ that induces an isomorphism of spectral sequences

$$E_r^{-p,n+1-q}(b) \cong {}'E_r^{-p,n+1-q}(b). \tag{4}$$

To obtain the vanishing of the E_∞ terms we look at the second spectral sequence. We have

$$''E_1^{i,-j}(b) = \mathcal{K}_{j,i-j+1}(K_Y \otimes \bigwedge^{b-i}\Sigma_{Y,L}, L).$$

Lemma 5.7 *Suppose that $''E_1^{i,-j}(b) = 0$ for all (i,j) such that $b-k+1 \leq i-j \leq b$. Then $E_\infty^{p,q}(b) = 0$ for all (p,q,b) such that $p+q+b \leq n+k$.*

Proof: It follows from the hypotheses of the Lemma that $''E_\infty^a(b) = 0$ for all $b-k+1 \leq a \leq b$. As the spectral sequences $''E_r^{p,q}(b)$ and $'E_r^{p,q}(b)$ converge to the same limit, we find that $'E_\infty^a(b) = 0$. Hence $'E_\infty^{-p,n+1-q}(b) = 0$ for all (p,q,b) such that $p+q+b \leq n+k$ and the proof is finished using the isomorphism (4). $\square$

We have finally reduced the proof to a statement about the vanishing of certain Koszul cohomology groups. By Proposition 5.6 it suffices to show that

$$H^1(Y, \bigwedge^{j+1} M_L \otimes K_Y \otimes \bigwedge^{b-i}\Sigma_{Y,L} \otimes L^{i-j}) = 0$$

for all (i,j) such that $b-k+1 \leq i-j \leq b$. Take exterior powers in the exact sequence (3) and twist by K_Y to obtain an exact sequence

$$0 \to \Omega_Y^{n+2-b+i} \to K_Y \otimes \bigwedge^{b-i}\Sigma_{Y,L} \to \Omega_Y^{n+1-b+i} \to 0.$$

Using this exact sequence we reduce to a vanishing statement with exterior powers of the cotangent bundle.

Recall that a coherent sheaf $\mathcal{F}$ on a polarised variety $(Y, \mathcal{O}_Y(1))$ is said to be m–regular if

$$H^i(Y, \mathcal{F}(m-i)) = 0 \quad \text{for all } i > 0.$$

Note that this definition depends on the choice of a polarisation on Y. The *Castelnuovo–Mumford regularity* of $\mathcal{F}$ is the number

$$m(\mathcal{F}) = \min\{m \in \mathbb{Z} | \mathcal{F} \text{ is } m\text{–regular}\}.$$

Let $m_i = m(\Omega^i_Y)$ be the regularity of Ω^i_Y. Following Paranjape we introduce the number

$$m_Y = \max\{m_i - i - 1 | 0 \leq i \leq \dim Y\}.$$

Note that this number is always nonnegative if Y is a projective variety, as $H^i(Y, \Omega^i_Y) \neq 0$.

Exercise 5.8 Show that $m_Y = 0$ if Y is a projective space and that $m_Y = 1$ if Y is a smooth quadric.

If $Y = \mathbb{P}^{n+1}$ and $L = \mathcal{O}_{\mathbb{P}}(1)$ then $M_L \cong \Omega^1_{\mathbb{P}}(1)$ is 1–regular. Using a lemma of M. Green [10] one can show (with some work) that $\bigwedge^k M_L$ is k–regular on Y.

Lemma 5.9

$$H^i(Y, \Omega^j_Y \otimes \bigwedge\nolimits^a M_L \otimes \mathcal{O}_Y(k)) = 0$$

if $i \geq 1$ *and* $k + i \geq m_j + a$.

Remark 5.10 The symmetrizer lemma from Lecture 3 follows from Proposition 5.6 and Lemma 5.9 (applied with $Y = \mathbb{P}^{n+1}$).

To finish the proof we have to sort out all the vanishing conditions of this type and to translate these into conditions on the degrees $(d_0, \ldots, d_r)$ of the hypersurfaces. In addition there are conditions coming from Proposition 5.6. These can be treated in a similar way at the cost of introducing stronger degree conditions. The final result is:

Theorem 5.11. *Let $(Y, \mathcal{O}_Y(1))$ be a smooth polarised variety of dimension $n+r+1$. Let $d_0, \dots, d_r$ be natural numbers ordered in such a way that $d_0 \geq \cdots \geq d_r$. Define $E = \mathcal{O}_Y(d_0) \oplus \dots \mathcal{O}_Y(d_r)$ and let $U \subset \mathbb{P}H^0(Y, E)$ be the complement of the discriminant locus. Let m_j be the regularity of Ω_Y^j and define*

$$m_Y = \max\{m_j - j - 1 | 0 \leq j \leq \dim Y\}.$$

Let $c \leq n$ be a nonnegative integer, and set $\mu = [\frac{n+c}{2}]$. Consider the conditions

(C) $\sum_{\nu=\min(c,r)}^{r} d_\nu \geq m_Y + \dim Y - 1;$

(C_i) $\sum_{\nu=i}^{r} d_\nu + (\mu - c + i)d_r \geq m_Y + \dim Y + c - i.$

If condition (C) is satisfied and if the conditions (C_i) are satisfied for all i with $0 \leq i \leq \min(c-1, r)$, then for every smooth morphism $g : T \to U$ we have $H^{n+k}(Y_T, X_T) = 0$ for all $k \leq c$.

In some cases we can avoid the extra conditions coming from Proposition 5.4. Suppose that

$$(*) \qquad H^i(Y, \Omega_Y^j(k)) = 0 \text{ for all } i > 0, j \geq 0, k > 0.$$

If Y satisfies this condition, we can omit the condition (C). Examples of varieties Y that satisfy condition $(*)$ are projective spaces and, more generally, smooth toric varieties. Condition $(*)$ also holds if Y is an abelian variety.

Bibliographical references. A good introduction to Koszul cohomology is [9]. A detailed proof of Theorem 5.11 can be found in [23, Thm. 3.13] (where the condition $c \leq n$ should be added in the statement of the theorem). In the case $Y = \mathbb{P}^N$ effective versions of Nori's theorem have been obtained by Voisin [29] and by M. Asakura and S. Saito [2].

6 Applications of Nori's theorem

We keep the notation of the previous section: let $(Y, \mathcal{O}_Y(1))$ be a smooth polarised variety and set $E = \mathcal{O}_Y(d_0) \oplus \dots \oplus \mathcal{O}_Y(d_r)$, $S = \mathbb{P}H^0(Y, E)$. As a first application of Nori's theorem, we mention the following result.

Proposition 6.1 *Let $X \subset Y$ be a complete intersection of multidegree $(d_0, \ldots, d_r)$ and dimension n. Suppose that Z is a cycle of codimension $p \leq n$ on Y such that $Z \cap X_s$ is rationally equivalent to zero for general $s \in S$. If $\min(d_0, \ldots, d_r) \gg 0$ then $\mathrm{cl}(Z) = 0$ in $H^{2p}(Y, \mathbb{Q})$.*

Proof: Let $X_S \subset Y_S$ be the universal family of complete intersections with inclusion map $r : X_S \to Y_S$. Set $Z_S = r^* p_Y^* Z \in \mathrm{CH}^p(X_S)$, and let η be the generic point of S. As we have (cf. the lectures of J. Lewis)

$$\mathrm{CH}^p(X_\eta) = \varinjlim_{U \subset S} \mathrm{CH}^p(X_U)$$

there exists a Zariski open subset $U \subset S$ such that $[Z_U] = 0$ in $\mathrm{CH}^p(X_U)$, hence $\mathrm{cl}(Z_U) = 0$. By Nori's theorem the restriction map

$$H^{2p}(Y_U, \mathbb{Q}) \to H^{2p}(X_U, \mathbb{Q})$$

is injective if $\min(d_0, \ldots, d_r) \gg 0$. Hence $\mathrm{cl}(p_Y^* Z) = 0$ and $\mathrm{cl}(Z) = 0$ because $p_Y^* : H^{2p}(Y, \mathbb{Q}) \to H^{2p}(Y_U, \mathbb{Q})$ is injective. □

For cycles of codimension $p < n$, Nori has shown that the result of Proposition 6.1 remains true under the weaker hypothesis that $Z \cap X_s$ is algebraically equivalent to zero. This result is a generalisation of Griffiths's theorem.

Theorem 6.2 (Nori) *(notation as in Proposition 6.1) Suppose that $p < n$ and that $Z \cap X_s$ is algebraically equivalent to zero for very general $s \in S$. If $\min(d_0, \ldots, d_r) \gg 0$ then $\mathrm{cl}_Y(Z) = 0$.*

Proof: Set $Z_S = r^* p_Y^* Z \in \mathrm{CH}^p(X_S)$. By the definition of algebraic equivalence, there exist a smooth morphism $T \to S$, a family of smooth curves $C_T \to T$, a relative divisor $D_T \in \mathrm{CH}^1_{\mathrm{hom}}(C_T/T)$ and a cycle $\Gamma \in \mathrm{CH}^p(C_T \times_T X_T)$ such that $Z_T = \Gamma_*(D_T)$. The map $\Gamma_* : \mathrm{CH}^1(C_T) \to \mathrm{CH}^p(X_T)$ is obtained from the diagram

$$\begin{array}{ccc} X_T \times_T C_T & \xrightarrow{p_2} & C_T \\ \Big\downarrow{\scriptstyle p_1} & & \\ X_T & & \end{array}$$

by setting $\Gamma_*(D) = (p_1)_*(p_2^* D.\Gamma)$. Set $T' = C_T$. As $T' \to T$ is a smooth morphism, it follows from Nori's theorem that $H^{2p+1}(Y_{T'}, X_{T'}) = 0$ if $p < n$

and $\min(d_0,\dots,d_r) \gg 0$. Hence there exists $\gamma \in H^{2p}(Y_{T'},\mathbb{Q})$ such that $r^*\gamma = \mathrm{cl}(\Gamma)$. Consider the diagram

$$\begin{array}{ccc} Y_T \times_T C_T = Y_{T'} & \xrightarrow{\pi_2} & T' = C_T \\ \downarrow{\scriptstyle \pi_1} & & \\ Y_T & & \end{array}$$

and put $\alpha = (\pi_1)_*(\pi_2^* D_T.\gamma) \in H^{2p}(Y_T,\mathbb{Q})$. By construction we have

$$r^*\alpha = r^* Z_T = r^* p_Y^* Z,$$

hence $\alpha = p_Y^* Z$ by Nori's theorem. If we restrict to the fiber over a point $t \in T$ we obtain

$$\mathrm{cl}(Z) = \pi_1(t)_*(\pi_2(t)^* D_t.\gamma|_{Y\times\{t\}}).$$

As $\mathrm{cl}(D_t) = 0$, it follows that $\mathrm{cl}(Z) = 0$. □

Remark 6.3 We can replace Betti cohomology by Deligne cohomology in the proof of Theorem 6.2 to obtain a stronger conclusion: if $Z \cap X_s$ is algebraically equivalent to zero for very general s, then the Deligne class $\mathrm{cl}_{\mathcal{D}}(Z) \in H^{2p}_{\mathcal{D}}(Y,\mathbb{Q}(p))$ is zero. Using this result, Albano and Collino [1] have shown that if $Y \subset \mathbb{P}^8$ is a general cubic sevenfold and if $X = Y \cap D_1 \cap D_2$ is a very general complete intersection of Y with two hypersurfaces of sufficiently large degree, then $\mathrm{Griff}^4(X) \otimes \mathbb{Q} \neq 0$ (they even show that this vector space is not finite dimensional). Note that the nonzero elements in the Griffiths group could not have been detected by the Abel–Jacobi map as $J^4(X) = 0$.

We have seen that Nori's theorem implies the theorems of Noether–Lefschetz and Green–Voisin. One obtains effective versions of these theorems using Theorem 5.11 (take $c = 0$ and $n = 2m$ resp. $c = 1$ and $n = 2m-1$). The degree bounds for these theorems are optimal if $Y = \mathbb{P}^{n+r+1}$; see [21].

Nori's theorem can also be used to study the regulator maps defined on Bloch's higher Chow groups.

Theorem 6.4 *Let X be a very general complete intersection of multidegree $(d_0,\dots,d_r)$ in Y. If $\min(d_0,\dots,d_r) \gg 0$ and $2p-k \leq 2n-1$, the image of the (rational) regulator map*

$$c_{p,k} : \mathrm{CH}^p(X,k)_{\mathbb{Q}} \to H^{2p-k}_{\mathcal{D}}(X,\mathbb{Q}(p))$$

is contained in the image of $i^* : H^{2p-k}_{\mathcal{D}}(Y,\mathbb{Q}(p)) \to H^{2p-k}_{\mathcal{D}}(X,\mathbb{Q}(p))$.

The proof of this Theorem is analogous to the proof of Theorem 4.3; it can be found in [20]. We consider the applications of this result to the higher Chow groups $CH^p(X,1)$ and $\mathrm{CH}^p(X,2)$. Recall that the group $\mathrm{CH}^p_{\mathrm{dec}}(X,1)$ of decomposable higher Chow cycles is defined as the image of the natural map

$$\mathrm{CH}^p(X)\otimes\mathbb{C}^*\to\mathrm{CH}^p(X,1).$$

The cokernel of this map is denoted by $\mathrm{CH}^p_{\mathrm{ind}}(X,1)$ (indecomposable higher Chow cycles). Similarly we define $H^{2p-1}_{\mathcal{D},dec}(X,\mathbb{Z}(p))$ as the image of the natural map

$$H^{2p-2}_{\mathcal{D}}(X,\mathbb{Z}(p-1))\otimes H^1_{\mathcal{D}}(X,\mathbb{Z}(1))\to H^{2p-1}_{\mathcal{D}}(X,\mathbb{Z}(p)).$$

A higher Chow cycle $z\in\mathrm{CH}^p(X,1)$ is said to be R–decomposable (regulator decomposable) if $c_{p,1}(z)\in H^{2p-1}_{\mathcal{D},dec}(X,\mathbb{Z}(p))$.

From Theorems 6.4 and 5.11 we obtain the following result.

Theorem 6.5. *Let $X=V(d_0,\ldots,d_r)\subset\mathbb{P}^{2m+r+1}$ be a smooth complete intersection of dimension $2m$ ($m\geq 1$, $d_0\geq\ldots\geq d_r$), $i:X\to\mathbb{P}^{2m+r+1}$ the inclusion map. If X is very general and if*

(C_0) $\sum_{i=0}^r d_i+(m-1)d_r\geq 2m+r+3$

(C_1) $\sum_{i=1}^r d_i+m\,d_r\geq 2m+r+2$

the image of the (rational) regulator map

$$c_{m+1,1}:\mathrm{CH}^{m+1}(X,1)_{\mathbb{Q}}\to H^{2m+1}_{\mathcal{D}}(X,\mathbb{Q}(m+1))$$

coincides with the image of the composed map ($N=2m+r+1$)

$$\mathrm{CH}^{m+1}(\mathbb{P}^N,1)_{\mathbb{Q}}\xrightarrow{\sim}H^{2m+1}_{\mathcal{D}}(\mathbb{P}^N,\mathbb{Q}(m+1))\to H^{2m+1}_{\mathcal{D}}(X,\mathbb{Q}(m+1)).$$

As $\mathrm{CH}^{m+1}(\mathbb{P}^N,1)\cong\mathbb{C}^*$ it follows that every element $z\in\mathrm{CH}^{m+1}(X,1)$ is R–decomposable modulo torsion. The exceptional cases include quartic surfaces and cubic fourfolds. For these cases the regulator map has been studied in [20] and [6]; in both cases the group of regulator indecomposable higher Chow cycles is nonzero, and it is not even finitely generated.

One can think of $\mathrm{CH}^p_{\mathrm{dec}}(X,1)$ as a kind of analogue of $\mathrm{CH}^p_{\mathrm{alg}}(X)$ for higher Chow cycles, and of $\mathrm{CH}^p_{\mathrm{ind}}(X,1)$ as an analogue of the Griffiths group. Collino [6] has proved an analogue of Griffiths's theorem for higher Chow cycles.

Theorem 6.6 *Let $(Y, \mathcal{O}_Y(1))$ be a smooth polarised variety and let X be a very general complete intersection of multidegree $(d_0, \ldots, d_r)$ and dimension $n \geq 2$. Let $Z \in \mathrm{CH}^p(Y, 1)$ be a higher Chow cycle such that $Z \cap X_s$ is decomposable for very general s. If $p \leq n$ and $\min(d_0, \ldots, d_r) \gg 0$ then Z is R–decomposable.*

Proof: After passing to a suitable covering $T \to S$ of the base space S of the universal family X_S, we may assume that $Z_T = r^* p_Y^* Z \in \mathrm{CH}^p_{\mathrm{dec}}(X_T, 1)$, hence $c_{p,1}(Z_T) \in H^{2p-1}_{\mathcal{D},dec}(X_T, \mathbb{Q}(p))$. Look at the commutative diagram

$$\begin{array}{ccc}
H^{2p-2}_{\mathcal{D}}(Y_T, \mathbb{Q}(p-1)) \otimes H^1_{\mathcal{D}}(Y_T, \mathbb{Q}(1)) & \to & H^{2p-1}_{\mathcal{D}}(Y_T, \mathbb{Q}(p)) \\
\downarrow & & \downarrow \\
H^{2p-2}_{\mathcal{D}}(X_T, \mathbb{Q}(p-1)) \otimes H^1_{\mathcal{D}}(X_T, \mathbb{Q}(1)) & \to & H^{2p-1}_{\mathcal{D}}(X_T, \mathbb{Q}(p)).
\end{array}$$

By Nori's theorem the vertical maps in this diagram are isomorphisms if $p \leq n$ and $\min(d_0, \ldots, d_r) \gg 0$. Hence $p_Y^* Z$ is R–decomposable, and by restricting to a fiber $Y \times \{t\}$ we find that Z is R–decomposable. $\square$

Remark 6.7 Collino applied this theorem to a cubic fourfold $Y \subset \mathbb{P}^5$ and obtained that $\mathrm{CH}^3_{\mathrm{ind}}(X_s, 1) \otimes \mathbb{Q}$ is nonzero for a very general hypersurface section $X_s \subset Y$ of sufficiently large degree. (He even showed that this vector space is infinite dimensional.)

For the higher Chow group $\mathrm{CH}^p(X, 2)$ we obtain the following result.

Theorem 6.8 *Let $X = V(d_0, \ldots, d_r) \subset \mathbb{P}^{2m+r}$ be a smooth complete intersection of dimension $2m - 1$ ($m \geq 1$, $d_0 \geq \ldots \geq d_r$), $i : X \to \mathbb{P}^{2m+r}$ the inclusion map. If X is very general and if*

(C_0) $\sum_{i=0}^{r} d_i + (m-1) d_r \geq 2m + r + 2$

(C_1) $\sum_{i=1}^{r} d_i + m\, d_r \geq 2m + r + 1$

the image of the (rational) regulator map

$$c_{m+1,2} : \mathrm{CH}^{m+1}(X, 2)_{\mathbb{Q}} \to H^{2m}_{\mathcal{D}}(X, \mathbb{Q}(m+1))$$

is zero.

Remark 6.9 The proof of Theorem 6.8 contains a small subtlety. To prove it we do not need the full strength of Nori's theorem, but the following result: if $F^{m+1}H^{2m+1}(Y_T, X_T) = 0$ and $F^m H^{2m}(Y_T, X_T) = 0$, then the restriction map

$$H^{2m}_{\mathcal{D}}(Y_T, \mathbb{Q}(m+1)) \to H^{2m}_{\mathcal{D}}(X_T, \mathbb{Q}(m+1))$$

is surjective. A proof of Theorem 6.8 for plane curves appeared in [5, (7.14)].

Example 6.10 Let $(Y, \mathcal{O}_Y(1))$ be a polarised abelian variety. As the tangent bundle T_Y is trivial, condition $(*)$ is satisfied by the Kodaira vanishing theorem. As Ω^m_Y is trivial for all m, the proof of Theorem 5.11 shows that we can substract $m_Y + \dim Y + 1$ on the right hand side of the inequality of condition (C_i). Hence condition (C_i) can be replaced by the weaker condition

(C'_i) $\sum_{\nu=i}^{r} d_\nu + (\mu - c + i)d_r \geq c - i - 1.$

This condition is empty if $c \leq 2$. For $c = 2$ we obtain a result on the Abel–Jacobi map for complete intersections in abelian varieties without degree conditions; see [23, Thm. 4.8].

Example 6.11 We consider an example mentioned in the introduction of Nori's paper. Let $Y \subset \mathbb{P}^7$ be a smooth quadric, and let $X = Y \cap V(d_0, d_1)$, $d_0 \geq d_1$, be a smooth complete intersection in Y. In this case condition $(*)$ is not satisfied. As $m_Y = 1$, Paranjape's results show that $H^{n+k}(Y_T, X_T) = 0$ for $k \leq 3$ if $d_1 \geq 9$. As $c = 3$ and $r = 1$, the conditions of Theorem 5.11 read (C) : $d_1 \geq 6$, (C_0) : $d_0 + d_1 \geq 10$, (C_1) : $2d_1 \geq 9$. Using precise vanishing theorems for the groups $H^i(Y, \Omega^j_Y(k))$, we find that the bound in condition (C) can be improved to $d_1 \geq 5$. By theorem 6.2 we obtain that $\mathrm{Griff}^3(X) \otimes \mathbb{Q} \neq 0$ if X is very general and $d_1 \geq 5$. Note that $\mathrm{Griff}^3(X) \otimes \mathbb{Q} = 0$ if $d_0 + d_1 < 6$ by a result of Bloch and Srinivas [3].

As we have seen, Nori's theorem cannot be applied to zero–cycles. Nevertheless it is possible to obtain results on zero cycles using Nori's techniques. As an example we mention a theorem of Voisin [28]. For a smooth variety X we write $A_0(X)$ for the Chow group of zero cycles of degree zero on X.

Theorem 6.12 (Voisin) *Let $S \subset \mathbb{P}^3$ be a very general surface of degree $d \geq 5$, and let $C \subset S$ be a smooth plane section with inclusion map $i : C \to S$. Then the kernel of the map*

$$i_* : A_0(C) \to A_0(S)$$

coincides with the subgroup $\mathrm{Tors}(A_0(C))$ *of torsion points of* $A_0(C)$.

Proof: Over the moduli space B of pairs (S, C) as above we have universal families C_B and S_B. By passing to a covering T of B we can spread out an element $z_0 \in \ker i_*$ to a relative cycle $Z_T \in A_0(C_T/T)$ such that $Z_T(t) \in \ker i_*$ for all $t \in T$ and such that $Z_T(t_0) = z_0$. As $Z(t)$ is rationally equivalent to zero for all $t \in T(\mathbb{C})$ we may assume that $r_* \operatorname{cl}_{\mathcal{D}}(Z_T) \in H^4_{\mathcal{D}}(S_T, \mathbb{Q})$ is zero, by replacing T by a suitable Zariski open subset of T. Set

$$U_T = \mathbb{P}^3_T \setminus \mathbb{P}^2_T, \quad V_T = S_T \setminus C_T.$$

Consider the commutative diagram

$$\begin{array}{ccccc}
H^3_{\mathcal{D}}(U_T, \mathbb{Q}(2)) & \to & H^3_{\mathcal{D}}(V_T, \mathbb{Q}(2)) & \to & H^4_{\mathcal{D}}(U_T, V_T, \mathbb{Q}(2)) \\
\downarrow r & & \downarrow & & \downarrow \\
H^2_{\mathcal{D}}(\mathbb{P}^2_T, \mathbb{Q}(1)) & \xrightarrow{k^*} & H^2_{\mathcal{D}}(C_T, \mathbb{Q}(1)) & \to & H^3_{\mathcal{D}}(\mathbb{P}^2_T, C_T, \mathbb{Q}(1)) \\
\downarrow & & \downarrow r_* & & \downarrow \\
H^4_{\mathcal{D}}(\mathbb{P}^3_T, \mathbb{Q}(2)) & \to & H^4_{\mathcal{D}}(S_T, \mathbb{Q}(2)) & \to & H^5_{\mathcal{D}}(\mathbb{P}^3_T, S_T, \mathbb{Q}(2)).
\end{array}$$

Nori's theorem cannot be applied to the groups $H^3(\mathbb{P}^2_T, C_T)$ and $H^5(\mathbb{P}^3_T, S_T)$, but using a variant of Nori's techniques we can show that

$$H^k(U_T, V_T) = 0 \quad \text{for all } k \leq 4 \text{ if } d \geq 5.$$

Combining this result with the vanishing of $H^k(U_T)$, we find that $H^k(V_T) = 0$ for all $k \leq 3$. Hence $H^3_{\mathcal{D}}(V_T, \mathbb{Q}(1)) = 0$ and

$$r_* : H^2_{\mathcal{D}}(C_T, \mathbb{Q}(1)) \to H^4_{\mathcal{D}}(S_T, \mathbb{Q}(2))$$

is injective. As $\operatorname{cl}_{\mathcal{D}}(Z_T) \in \ker r_*$ we get $\operatorname{cl}_{\mathcal{D}}(Z_T) = 0$. By restriction to the fiber over t_0 we see that $\operatorname{cl}_{\mathcal{D}}(z_0) \in H^2_{\mathcal{D}}(C_0, \mathbb{Q}(1))$ is zero. As $H^2_{\mathcal{D}}(C_0, \mathbb{Q}(1)) \cong \mathrm{Pic}(C_0) \otimes \mathbb{Q}$ it follows that $z_0 \in \mathrm{Tors}(A_0(C_0))$. The inclusion $\mathrm{Tors}(A_0(C)) \subseteq \ker i_*$ follows from Roitman's theorem. □

Remark 6.13 It is possible to prove a similar result for curves that are obtained by intersecting an ample divisor Y in a threefold W with a surface S that varies in a sufficiently ample linear system on W; see [22].

A Deligne cohomology

We recall the definition of the Deligne and Deligne–Beilinson cohomology groups.

Definition A.1 *Let X be a smooth projective variety. Set $\mathbb{Z}(p) = (2\pi i)^p\mathbb{Z}$. The p–th Deligne complex on X is the complex*

$$\mathbb{Z}_{\mathcal{D}}(p) = (\mathbb{Z}(p) \to \mathcal{O}_X \to \Omega^1_X \to \ldots \to \Omega^{p-1}_X)$$

concentrated in degrees $0, \ldots, p$. The hypercohomology group $\mathbb{H}^k(\mathbb{Z}_{\mathcal{D}}(p))$ is called the k–th Deligne cohomology group of X with coefficients in $\mathbb{Z}(p)$ and is denoted by $H^k_{\mathcal{D}}(X, \mathbb{Z}(p))$.

Example A.2 The exponential sequence shows that $\mathbb{Z}_{\mathcal{D}}(1) \simeq \mathcal{O}^*_X$, hence $H^k_{\mathcal{D}}(X, \mathbb{Z}(1)) \cong H^{k-1}(X, \mathcal{O}^*_X)$.

Definition A.3 *Let $f : A^\bullet \to B^\bullet$ be a morphism of complexes. The cone complex associated to f is the complex $C^\bullet(f) = B^\bullet[-1] \oplus A^\bullet$ with differential $d_C(b, a) = (d_B(b) + f(a), -d_A(a))$. This complex fits into an exact sequence*

$$0 \to B^\bullet[-1] \to C^\bullet(f) \to A^\bullet \to 0.$$

We say that two complexes $A^\bullet$ and $B^\bullet$ are *quasi–isomorphic* if they have the same cohomology groups; notation $A^\bullet \simeq B^\bullet$.

Exercise A.4 Let $f : A^\bullet \to B^\bullet$ be a morhpism of complexes. Write $K^\bullet = \ker f$, $Q^\bullet = \operatorname{coker} f$.

(i) If f is surjective, then $C^\bullet(f) \simeq K^\bullet$;

(ii) if f is injective, then $C^\bullet(f) \simeq Q^\bullet[-1]$;

(iii) let $\bar{B}^\bullet$ be the quotient of $B^\bullet$ by a subcomplex $C^\bullet$ with inclusion map $i : C^\bullet \to B^\bullet$, and let $p : B^\bullet \to \bar{B}^\bullet$ be the projection map. Then $C^\bullet(p \circ f) \simeq C^\bullet(f - i)$.

The de Rham complex is filtered by subcomplexes

$$\sigma_{\geq p}\Omega_X^\bullet = (\Omega_X^p \to \Omega_X^{p+1} \to \dots)$$

that start in degree p (this filtration is called 'filtration bête' or stupid filtration). The quotient of $\Omega_X^\bullet$ by $\sigma_{\geq p}\Omega_X^\bullet$ is denoted by $\sigma_{<p}\Omega_X^\bullet$. As the Deligne complex fits into an exact sequence

$$0 \to \sigma_{<p}\Omega_X^\bullet[-1] \to \mathbb{Z}_{\mathcal{D}}(p) \to \mathbb{Z}(p) \to 0$$

we deduce from the previous exercise that

$$\mathbb{Z}_{\mathcal{D}}(p) \simeq \mathrm{Cone}(\mathbb{Z}(p) \oplus \sigma_{\geq p}\Omega_X^\bullet \to \Omega_X^\bullet).$$

Hence the Deligne cohomology groups fit into a long exact sequence

$$H^{k-1}(X,\mathbb{C}) \to H^k_{\mathcal{D}}(X,\mathbb{Z}(p)) \to H^k(X,\mathbb{Z}(p)) \oplus F^pH^k(X,\mathbb{C}) \to H^k(X,\mathbb{C}).$$

If $k = 2p$ we obtain a short exact sequence

$$0 \to J^p(X) \to H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p)) \to \mathrm{Hdg}^p(X) \to 0.$$

There exists a *Deligne cycle class map*

$$\mathrm{cl}^p_{\mathcal{D}} : \mathrm{CH}^p(X) \to H^{2p}_{\mathcal{D}}(X,\mathbb{Z}(p))$$

whose restriction to $\mathrm{CH}^p_{\mathrm{hom}}(X)$ coincides with the Abel–Jacobi map [8]. More generally there exist *regulator maps*

$$c_{p,k} : \mathrm{CH}^p(X,k) \to H^{2p-k}_{\mathcal{D}}(X,\mathbb{Z}(p))$$

on Bloch's higher Chow groups $\mathrm{CH}^p(X,k)$ that coincide with the Deligne cycle class map if $k = 0$.

For quasi–projective varieties there exists a variant of Deligne cohomology, called Deligne–Beilinson cohomology. Given a quasi–projective variety X, choose a good compactification $j : X \to \bar{X}$ with boundary $D = X \setminus \bar{X}$, and choose injective resolutions $\mathcal{I}^\bullet$ of $\mathbb{Z}_X(p)$ and $\mathcal{J}^\bullet$ of $\Omega_X^\bullet$. Put

$$Rj_*\mathbb{Z}_X(p) = j_*\mathcal{I}^\bullet, \quad Rj_*\Omega_X^\bullet = j_*\mathcal{J}^\bullet.$$

There is a natural map of complexes

$$\alpha : \sigma_{\geq p}\Omega_{\bar{X}}^\bullet(\log D) \oplus Rj_*\mathbb{Z}_X(p) \to Rj_*\Omega_X^\bullet.$$

The Deligne–Beilinson cohomology groups are defined as the hypercohomology groups of the complex $C^\bullet(\alpha)$. As before they fit into an exact sequence

$$H^{k-1}(X,\mathbb{C}) \to H^k_{\mathcal{D}}(X,\mathbb{Z}(p)) \to H^k(X,\mathbb{Z}(p)) \oplus F^pH^k(X,\mathbb{C}) \to H^k(X,\mathbb{C})$$

where $F^\bullet$ denotes the Hodge filtration of the mixed Hodge structure on $H^k(X,\mathbb{C})$.

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