The generalized Hodge conjecture for the quadratic complex of lines in projective four-space

J. Nagel

1 Introduction

Let X be a smooth projective variety defined over $\mathbb{C}$. The generalized Hodge conjecture $\mathrm{GHC}(X, 2p-1, p-1)$ (as corrected by Grothendieck) asserts that every $\mathbb{Q}$–sub Hodge structure $V \subset H^{2p-1}(X)$ of level one is supported in codimension $p-1$, i.e., if $V \subset F^{p-1}H^{2p-1}(X, \mathbb{C}) \cap H^{2p-1}(X, \mathbb{Q})$ then there should exist a subvariety $Z \subset X$ of codimension $p-1$ such that $V \subset \ker H^{2p-1}(X) \to H^{2p-1}(X \setminus Z)$. Let $J^p_{\mathrm{max}}(X)$ be the abelian subvariety of the intermediate Jacobian $J^p(X)$ that is associated to the maximal sub Hodge structure of level one contained in $H^{2p-1}(X)$, and let $J^p_{\mathrm{alg}}(X) = \psi_X(\mathrm{CH}^p_{\mathrm{alg}}(X))$ be the image of the Chow group of codimension p cycles algebraically equivalent to zero under the Abel–Jacobi map. One has $J^p_{\mathrm{alg}}(X) \subset J^p_{\mathrm{max}}(X)$, and $\mathrm{GHC}(X, 2p-1, p-1)$ is true if and only if equality holds; see [Mu, Lemma 4.3].

Miyaoka [Mi] has proved that every smooth threefold X of Kodaira dimension $\kappa(X) = -\infty$ is uniruled; hence $\mathrm{GHC}(X, 3, 1)$ holds by a Remark of Steenbrink [St, Prop (2.6)]. The conujecture $\mathrm{GHC}(X, 3, 1)$ has been verified for Fermat hypersurfaces of degree ≤ 10 in $\mathbb{P}^4$ [Sh], for the very general member of some families of threefolds with trivial canonical bundle (see [Bar 1] and [Ba 2]) and for the very general member of some families of threefolds of general type [Ros].

Some higher–dimensional examples where $\mathrm{GHC}(X, 2p-1, p-1)$ holds are (X general):

(i) $X = V(2,2) \subset \mathbb{P}^{2p+1}$, $X = V(2,2,2) \subset \mathbb{P}^{2p+2}$; cf. [Re], [T]

(ii) $(p = 3)$ $X = V(3) \subset \mathbb{P}^6$; see [C].

(iii) $X = V(1,1,1) \subset G(2, p+3)$; see [Don].

(iv) $(p = 4)$ $X = V(1,1) \subset G(3,6)$; see [Don].

Among the Fano threefolds is tha quadratic complex of lines in $\mathbb{P}^3$, which can be represented as an intersection of two quadrics in $\mathbb{P}^5$. The geometry of this variety has been studied extensively; see e.g.[GH, Chapter 6]. In this paper we consider the quadratic complex of lines in $\mathbb{P}^4$. This variety is a Fano fivefold X of index 3 whose cohomology group $H^5(X)$ carries a Hodge structure of level one with $h^{2,3}(X) = 10$; its geometry has been studied by the classical geometers B. Segre, J. Semple and L. Roth (cf. Remarks 2.7 and 3.9). We prove that $\mathrm{GHC}(X, 5, 2)$ holds if $X = V(2) \subset G(2,5)$ is general.

In Section 2 we show that for general X the Fano variety F_X of two–planes contained in X is a smooth curve, and we compute its numerical invariants. The surjectivity of the Abel–Jacobi mapping associated to this family of two–planes is shown in Section 3.

The main motivation for the consideration of this example was a general result, which states that for very general complete intersections of sufficiently high multidegree in Grassmann varieties the image of the Abel–Jacobi map is, up to torsion, completely determined by the group of primitive Hodge classes of the Grassmann variety; cf. Remark 3.9. This paper is a revised version of the last chapter of my thesis. I would like to thank S. Müller–Stach, J.P. Murre and C. Peters for helpful discussions.

2 The family of planes

Let V be a complex vector space of dimension 5, and let $G = G(2, V)$ be the Grassmann variety of lines in $\mathbb{P}^4 = \mathbb{P}(V)$. The variety G is embedded as a smooth six–dimensional subvariety of degree 5 in $\mathbb{P}^9 = \mathbb{P}(\wedge^2 V)$ by the Plücker embedding. We denote the line in $\mathbb{P}^4$ corresponding to a point $x \in G$

by ℓ_x. A quadratic line complex in G is the intersection of G with a quadric $Q \subset \mathbb{P}^9$; it corresponds to a five–dimensional family of lines in $\mathbb{P}^4$.

Let $p \in \mathbb{P}(V)$ be a point, and let $\sigma(p) = \{x \in G : p \in \ell_x\}$ be the corresponding Schubert cycle. Since the tangent space T_xG is spanned by

$$T_xG \cap G = \{z \in G : \ell_z \cap \ell_x \neq \emptyset\} = \cup_{p \in \ell_x} \sigma(p),$$

the line spanned by two points x, $y \in G$ is contained in G if and only if $\ell_x \cap \ell_y \neq \emptyset$. Hence G contains two families of 2–planes: the σ–planes (solid point–stars) and the ρ–planes (ruled planes) (cf. [SR, X, §4]). Let $h \subset \mathbb{P}^4$ be a hyperplane, let $p \in h$ be a point and let $w_2 \subset \mathbb{P}^4$ be a 2–plane. The σ–planes are the Schubert cycles $\sigma(p, h) = \{x \in G : p \in \ell_x \subset h\}$; the ρ–planes are the Schubert cycles $\sigma(w_2) = \{x \in G : \ell_x \subset w_2\}$.

Let $D(a_1, \ldots, a_k, n)$ be the flag variety of type $(a_1, \ldots, a_k, n)$, i.e., the variety that parametrizes flags of linear subspaces

$$V_{a_1} \subset V_{a_2} \subset \ldots \subset V_{a_k} \subset W,$$

where W is a complex vector space of dimension n and dim $V_i = i$. Instead of $D(a_1, \ldots, a_k, n)$ we sometimes write $D(a_1, \ldots, a_k, W)$.

The flag variety $D = D(a_1, \ldots, a_k, n)$ carries a sequence of universal subbundles

$$H_{a_1} \subset H_{a_2} \subset \ldots H_{a_k} \subset H_n = W \otimes_{\mathbb{C}} \mathcal{O}_D.$$

Let $H_{i,j} = H_i/H_j$ $(i > j)$ be the induced quotient bundles. The exact sequence $0 \to H_j \to H_i \to H_{i,j} \to 0$ is obtained by pulling back the tautological exact sequence on the Grassmann variety $G(a_j, a_i)$ via the projection map

$$p_{i,j} : D(a_1, \ldots, a_k, n) \to G(a_i, a_j).$$

The family of σ–planes on G is parametrized by the 7–dimensional flag variety $D = D(1, 4, 5)$; the family of ρ–planes on G is parametrized by the 6–dimensional flag variety $D(3, 5)$. In the sequel we shall concentrate on the family of σ–planes on G. The Plücker embedding $i : G(2, 5) \to \mathbb{P}^9$ sends a two–dimensional linear subspace $V_2 = \langle v_1, v_2 \rangle$ to the line in $\bigwedge^2 V$ spanned by $v_1 \wedge v_2$. A coordinate–free description of the Plücker embedding is

$$\begin{array}{rcl} i : G(2, V) & \longrightarrow & \mathbb{P}(\bigwedge^2 V) \\ (V_2, V) & \mapsto & (\bigwedge^2 V_2, \bigwedge^2 V). \end{array}$$

Note that i is an embedding because the pair $(W, \bigwedge^2 V) \in i(G)$ uniquely determines V_2 by

$$V_2 = \{v \in V : v \wedge w = 0 \text{ for all } w \in W\}.$$

Given a point $(V_1, V_4, V) \in D(1,4,5)$, we denote by $V_1 \bigwedge V_4$ the subspace of $\bigwedge^2 V$ spanned by the vectors $v \wedge w$, where $v \in V_1$ and $w \in V_4$. The Plücker embedding induces an embedding of the flag variety $D(1,4,5)$ into the Grassmann variety $G' = G(3,10)$ of 2–planes in $\mathbb{P}^9$: choose a vector v that spans V_1 and a basis $\{v, v_1, v_2, v_3\}$ for V_4, and map the point (V_1, V_4, V) to the 3–dimensional linear subspace of $\bigwedge^2 V$ spanned by $\{v \wedge v_1, v \wedge v_2, v \wedge v_3\}$. A coordinate–free description of this map is

$$\begin{aligned} j : D = D(1,4,5) &\rightarrow G(3,10) \\ (V_1, V_4, V) &\mapsto (V_1 \bigwedge V_4, \bigwedge^2 V). \end{aligned}$$

Note that we can recover the pair (V_1, V_4) from $(W, \bigwedge^2 V) \in \operatorname{im} j$ by setting

$$\begin{aligned} V_1 &= \{v \in V : v \wedge w = 0 \text{ for all } w \in W\} \\ V_4 &= \{v \in V : v \wedge w = 0 \text{ for some } w \in W\}. \end{aligned}$$

Let $X = G \cap Q$ be a quadratic line complex. The quadric Q corresponds to a symmetric form $Q \in S^2(\bigwedge^2 V^\vee)$. Let

$$0 \rightarrow \mathcal{S}_3 \rightarrow \bigwedge^2 V \otimes \mathcal{O}_{G'} \rightarrow \mathcal{Q}_7 \rightarrow 0$$

be the tautological exact sequence on $G' = G(3,10)$. This sequence induces a surjective map of vector bundles

$$S^2(\bigwedge^2 V^\vee) \otimes \mathcal{O}_{G'} \rightarrow S^2 \mathcal{S}_3^\vee$$

whose kernel we denote by K. Let $s : S^2(\bigwedge^2 V^\vee) \rightarrow H^0(G', S^2 \mathcal{S}_3^\vee)$ be the induced map on global sections. The Fano variety F_X of σ–planes contained in X is the zero scheme of the section $s(Q)$. Let

$$0 \rightarrow j^* K \rightarrow S^2(\bigwedge^2 V^\vee) \otimes \mathcal{O}_D \rightarrow j^* S^2 \mathcal{S}_3^\vee \rightarrow 0$$

be the exact sequence obtained by pullback to D. By composition of the inclusion map $\mathbb{P}(j^* K) \subset \mathbb{P}(S^2 \bigwedge^2 V^\vee) \times D$ and projection onto the first factor, we obtain a map

$$\mathbb{P}(j^* K) \rightarrow \mathbb{P}(S^2 \bigwedge^2 V^\vee)$$

that exhibits the projective bundle $\mathbb{P}(j^*K)$ as the universal family of Fano schemes of σ–planes over the family of quadratic line complexes (cf. [AK]).

To calculate the numerical invariants of the Fano scheme F_X, we determine the Chern classes of $j^*\mathcal{S}_3^\vee$.

Lemma 2.1. $j^*\mathcal{S}_3 = H_1 \otimes H_{4,1}$.

Proof: The fiber of $j^*\mathcal{S}_3$ over a point $x = (V_1, V_4, V)$ is $V_1 \bigwedge V_4$. Since the natural map $V_1 \bigwedge V_4 \to V_1 \otimes (V_4/V_1)$ is a canonical isomorphism, we obtain the desired isomorphism of vector bundles. $\square$

Remark 2.2. The previous result, whose original proof was simplified by a suggestion of L. Manivel, gives a method to compute the numerical invariants of the Fano schemes $F_k(X)$ of k–planes contained in X. It simplifies the method of computation used in [Ma].

The flag variety $D = D(1,4,5)$ is the incidence correspondence in $\mathbb{P}^4 \times (\mathbb{P}^4)^\vee$ with projections $p : D \to G(4,5) = (\mathbb{P}^4)^\vee$ and $q : D \to \mathbb{P}^4$. Note that $j^*\mathcal{S}_3 = H_1 \otimes H_{4,1} = q^*(\mathcal{Q}_{\mathbb{P}^4}(-1))$. To describe the Chow ring $\mathrm{CH}^*(D)$, we note that the projection p gives D the structure of a projective bundle $\mathbb{P}(\mathcal{S}_4)$ over $G(4,5)$. Set $x = c_1(\mathcal{O}_D(1))$ and $h = c_1(\mathcal{S}_4^\vee)$. The Chow ring of D is

$$CH^*(D) \cong \mathbb{Z}[x,h]/(x^4 - hx^3 + h^2x^2 - h^3x + h^4, h^5).$$

The first Chern classes of the universal bundles $H_1 = q^*\mathcal{O}_{\mathbb{P}^4}(-1)$ and $H_4 = p^*\mathcal{S}_4$ are $c_1(H_1) = -x$, $c_1(H_4) = -h$. Using the exact sequence

$$0 \to H_{4,1}^\vee \to H_4^\vee \to H_1^\vee \to 0$$

we compute the Chern polynomial of $H_{4,1}^\vee$:

$$\begin{aligned} c(H_{4,1}^\vee) &= (1 + ht + h^2t^2 + h^3t^3 + h^4t^4)(1 + xt)^{-1} \\ &= 1 + (h-x)t + (h^2 - hx + x^2)t^2 + (h^3 - h^2x + hx^2 - x^3)t^3. \end{aligned}$$

Using Lemma 2.1, we find that the Chern classes of $j^*\mathcal{S}_3^\vee$ are

$$\begin{aligned} c_1(j^*\mathcal{S}_3^\vee) &= 3c_1(H_1^\vee) + c_1(H_{4,1}^\vee) = h + 2x \\ c_2(j^*\mathcal{S}_3^\vee) &= 3c_1(H_1^\vee)^2 + 2c_1(H_1^\vee)c_1(H_{4,1}^\vee) + c_2(H_{4,1}^\vee) \\ &= 2x^2 + hx + h^2 \\ c_3(j^*\mathcal{S}_3^\vee) &= c_1(H_1^\vee)^3 + c_1(H_1^\vee)^2c_1(H_{4,1}^\vee) + c_1(H_1^\vee)c_2(H_{4,1}^\vee) + c_3(H_{4,1}^\vee) \\ &= hx^2 + h^3. \end{aligned}$$

The top Chern class of $E = S^2(j^*\mathcal{S}_3^\vee)$ is

$$\begin{aligned} c_6(E) &= 8c_1(j^*\mathcal{S}_3^\vee)c_2(j^*\mathcal{S}_3^\vee)c_3(j^*\mathcal{S}_3^\vee) - 8c_3(j^*\mathcal{S}_3^\vee)^2 \\ &= 32hx^5 + 24h^2x^4 + 56h^3x^3 + 24h^4x^2 + 24h^5x \\ &= 80h^3x^3. \end{aligned}$$

Let $\pi : \mathcal{X} \to \mathbb{P}H^0(\mathbb{P}^9, \mathcal{O}_{\mathbb{P}^9}(2))$ be the universal family of quadratic line complexes. Set $X_t = \pi^{-1}(t)$.

Lemma 2.3. *If $X \subset G$ is a general quadratic line complex, then F_X is a smooth curve of genus 161.*

Proof: Consider the universal family of Fano schemes

$$p : \mathbb{P}(j^*K) \to \mathbb{P}(S^2{\bigwedge}^2 V^\vee).$$

Note that $p^{-1}(t) = F_{X_t} = D \cap F_2(Q_t)$, where $F_2(Q_t)$ is the Fano variety of 2–planes contained in the quadric Q_t. For a general $Q \in \mathbb{P}(\bigwedge^2 S^2V^\vee)$ we shall compute the intersection $[D].[F_2(Q)] \in \mathrm{CH}^{20}(G')$, $G' \cong G(3,10)$. Because $[F_2(Q)] = c_6(S^2\mathcal{S}_3^\vee)$, we have

$$\begin{aligned} [D].[F_2(Q)] &= j_*(j^*[F_2(Q)]) \\ &= j_*c_6(E) \\ &= j_*(80h^3x^3). \end{aligned}$$

The projection formula shows that

$$\begin{aligned} j_*(80h^3x^3).c_1(\mathcal{S}_3^\vee) &= j_*(80h^3x^3.j^*c_1(\mathcal{S}_3^\vee)) \\ &= j_*(80h^3x^3.(2x+h)) \\ &= 240, \end{aligned}$$

where we have used that $h^3x^4 = h^4x^3$. Hence $[D].[F_2(Q)] = j_*(80h^3x^3) \neq 0$ and $D \cap F_2(Q) \neq \emptyset$ for general Q by Kleiman's transversality theorem [HAG, III, Thm. 10.8]. It follows that the map p is dominant, and hence surjective. As $\mathbb{P}(j^*K)$ is a smooth and irreducible variety of dimension 55, the general fiber F_X is a smooth curve by generic smoothness [HAG, III, Cor. 10.7]. The genus of F_X, for general X, is computed using the exact sequences

$$0 \to T_{F_X} \to T_D|_{F_X} \to E|_{F_X} \to 0 \tag{1}$$

$$0 \to T_v \to T_D \to p^*T_{G(4,5)} \to 0 \tag{2}$$

$$0 \to \mathcal{O}_D \to p^*\mathcal{S}_4 \otimes \mathcal{O}_D(1) \to T_v \to 0. \tag{3}$$

From the sequences (2) and (3) we obtain

$$c_1(T_D) = c_1(T_v) + c_1(p^* T_{G(4,5)}) = 4x - h + 5h = 4x + 4h.$$

Let $j_X : F_X \to D$ be the inclusion map. The exact sequence (1) shows that

$$\begin{aligned}(j_X)_* c_1(F_X) &= (c_1(T_D) - c_1(S^2 \mathcal{S}_3^\vee)).[F_X] \\ &= (4x + 4h - 4(2x + h)).80h^3x^3 \\ &= -320h^3x^4 = -320h^4x^3,\end{aligned}$$

hence $2 - 2g(F_X) = -320$. □

Since the vector bundle E is not ample, (cf. Remark 3.2), it is not clear whether the curve F_X is connected. To show that F_X is connected, we calculate the cohomology of the exterior powers of $E^\vee$ on the flag variety D. We refer to [FH] and [Hu] for basic facts concerning representation theory. Let G be a connected and simply connected complex Lie group, and let $P \subset G$ be a parabolic subgroup. The quotient space $Y = G/P$ is a compact homogeneous space.

Let R^+ be the finite set of positive roots, and let $T \subset G$ be a maximal torus. Let B be the Borel subgroup generated by T and the negative root groups. The Killing form induces an inner product $(,)$ on the character group $\Lambda = \mathrm{Hom}(T, \mathbb{C}^*)$. A weight $\lambda_i n \Lambda$ is called *singular* if $(\lambda, \alpha) = 0$ for some positive root $\alpha \in R^+$. If λ is not singular, it is called *regular* and we define

$$\mathrm{index}(\lambda) = \#\{\alpha \in R^+ : (\lambda, \alpha) < 0\}.$$

The cohomology groups of irreducible homogeneous vector bundles, i.e., vector bundles that are induced by irreducible representations of P, can be computed by the following theorem of Bott (see [Bott, Theorem IV']):

Theorem 2.4 (Bott) *Let P be a parabolic subgroup of a semisimple complex Lie group G. Let W_λ be the irreducible P–module with highest weight λ and let $E_\lambda = G \times_P W_\lambda$ the corresponding homogeneous vector bundle on $Y = G/P$. Let $\delta = \sum_i \lambda_i$ be the sum of the fundamental dominant weights, and let W be the Weyl group.*

(i) If $\lambda + \delta$ is singular, then $H^p(Y, E_\lambda) = 0$ for all $p \geq 0$.

(ii) If $\lambda+\delta$ is regular, then

$$H^p(Y,E_\lambda)=\begin{cases}0 & \text{if } p\neq \operatorname{index}(\lambda+\delta)\\ \Gamma_{\mu-\delta} & \text{if } p=\operatorname{index}(\lambda+\delta),\end{cases}$$

where μ is the unique dominant weight in the W–orbit of $\lambda+\delta$ and $\Gamma_{\mu-\delta}$ denotes the irreducible G–module with highest weight $\mu-\delta$.

Choose a basis $\{e_1,\dots,e_5\}$ for V, and let $W\subset V$ be the subspace spanned by e_2, e_3 and e_4. Let $U\subset V$ be the one–dimensional subspace spanned by e_5. The flag variety D is a homogeneous space of the form $D=\mathrm{SL}(5,\mathbb{C})/P$, where

$$P=\left\{\begin{pmatrix}h_1 & 0 & 0\\ h_2 & h_3 & 0\\ h_4 & h_5 & h_6\end{pmatrix}: h_1,\ h_6\in\mathbb{C}^*,\ h_3\in\mathrm{GL}(3,\mathbb{C}), h_1.\det(h_3).h_6=1\right\}.$$

Let $\rho:P\to W$ be the representation of P defined by $\rho(h)=h_3$, and let $\chi:P\to U$ be the character $\chi(h)=h_6$. The homogeneous vector bundle $j^*\mathcal{S}_3$ corresponds to the irreducible representation $\rho\otimes\chi:P\to W\otimes U$. Since

$$\begin{aligned}S^2(W\otimes U) &= S^2W\otimes U^{\otimes 2}\\ \textstyle\bigwedge^m S^2(W\otimes U) &= \textstyle\bigwedge^m(S^2W)\otimes U^{\otimes 2m},\end{aligned}$$

it suffices to determine the highest weights of the representations $\bigwedge^m(S^2W)$ for $1\le m\le 6$.

The representation ρ is induced by the standard representation of the semisimple part $P_{\mathrm{ss}}\cong\mathrm{SL}(3,\mathbb{C})$. The irreducible representation of $\mathrm{SL}(3,\mathbb{C})$ with highest weight $(\beta_2,\beta_3,\beta_4)=\beta_2e_2+\beta_3e_3+\beta_4e_4$ is denoted by $\Gamma_{\beta_2,\beta_3,\beta_4}$.

Lemma 2.5. *The decompositions of the exterior powers $\bigwedge^k(S^2W)$ into irreducible representations of $\mathrm{SL}(3,\mathbb{C})$ are*

$$\begin{aligned}S^2W &\cong \Gamma_{2,0,0} & \textstyle\bigwedge^4(S^2W)&\cong\Gamma_{4,3,1}\\ \textstyle\bigwedge^2(S^2W)&\cong\Gamma_{3,1,0} & \textstyle\bigwedge^5(S^2W)&\cong\Gamma_{4,4,2}\\ \textstyle\bigwedge^3(S^2W)&\cong\Gamma_{4,1,1}\oplus\Gamma_{3,3,0} & \textstyle\bigwedge^6(S^2W)&\cong\Gamma_{4,4,4}.\end{aligned}$$

Proof: This follows either from direct computation of the weights or by applying Formula (2.6) in [JPW]. □

Note that a weight $\lambda = (\beta_1, \dots, \beta_5)$ of $\mathrm{SL}(5,\mathbb{C})$ is singular if and only if there exist indices $1 \leq i < j \leq 5$ such that $\beta_i = \beta_j$. The index of $\lambda = (\beta_1, \dots, \beta_5)$ is

$$\begin{aligned} \operatorname{index}(\lambda) &= \#\{\alpha \in R^+ : (\lambda, \alpha) < 0\} \\ &= \#\{(i,j) : 1 \leq i < j \leq 5,\ \beta_i < \beta_j\}. \end{aligned}$$

Let $\delta = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 4e_1 + 3e_2 + 2e_3 + e_4$ be the sum of the fundamental dominant weights.

Using Lemma 2.5, we make a table of the highest weights λ_i associated to the vector bundles $\bigwedge^k E^\vee$ and the indices of $\lambda_i + \delta$ (if the weight is singular, we put a bar). Note that the highest weight of an irreducible representation of P is dominant for the semisimple part $P_{\mathrm{ss}} \cong \mathrm{SL}(3,\mathbb{C})$ of P. To emphasize this we write $\beta_1 e_1 + \ldots + \beta_5 e_5 = (\beta_1; \beta_2, \beta_3, \beta_4; \beta_5)$.

	λ_i	$\operatorname{index}(\lambda_i + \delta)$
$E^\vee$	(0;2,0,0;2)	-
$\bigwedge^2 E^\vee$	(0;3,1,0;4)	-
$\bigwedge^3 E^\vee$	(0;4,1,1;6)	4
	(0;3,3,0;6)	-
$\bigwedge^4 E^\vee$	(0;4,3,1;8)	6
$\bigwedge^5 E^\vee$	(0;4,4,2;10)	6
$\bigwedge^6 E^\vee$	(0;4,4,4;12)	7

Lemma 2.6. *If $X \subset G$ is a general quadratic line complex, the curve F_X is connected.*

Proof: In Lemma 2.3 we showed that F_X is a smooth curve. Since F_X is the zero locus of the global section $s(Q) \in H^0(D, E)$, we have a Koszul resolution

$$0 \to \bigwedge^6 E^\vee \to \cdots \to \bigwedge^2 E^\vee \to E^\vee \to \mathcal{O}_D \to \mathcal{O}_{F_X} \to 0$$

for $\mathcal{O}_{F_X}$. Hence $H^0(F_X, \mathcal{O}_{F_X}) \cong H^0(D, \mathcal{O}_D) = \mathbb{C}$ if $H^p(D, \bigwedge^p E^\vee) = 0$ for $1 \leq p \leq 6$. This follows from Theorem 2.4, as the weights λ_i associated to $\bigwedge^p E^\vee$ are either singular or have $\operatorname{index}(\lambda_i + \delta) \neq p$. □

Remark 2.7. The quadratic complex of lines in $\mathbb{P}^4$ has been studied from a different point of view by B. Segre [Seg]. He considers the Fano variety $F_3(G)$ of 3–planes on $G(2,5)$. Since every 3–plane contained in G is a Schubert cycle $\sigma(p)$ of lines through a point $p \in \mathbb{P}^4$, $F_3(G)$ is isomorphic to $\mathbb{P}^4$. A point $p \in \mathbb{P}^4$ is called singular (with respect to X) if the corresponding 3–plane $\sigma(p)$ is tangent to the quadric $Q \subset \mathbb{P}^9$ that defines X. For a general quadratic line complex X, Segre claims the following results:

1. The singular points are parametrized by a sextic hypersurface $\Sigma \subset \mathbb{P}^4$.

2. The points $p \in \mathbb{P}^4$ such that $\operatorname{rank}(Q|_{\sigma(p)}) \leq 2$ (i.e., the restriction of Q to $\sigma(p)$ is a union of two planes) are parametrized by a smooth curve $C \subset \Sigma$ of degree 40 and genus 81.

To rephrase these results in modern language, we consider the map

$$\begin{array}{rcl} g : \mathbb{P}^4 & \rightarrow & G(4, \bigwedge^2 V) \\ (V_1, V_5) & \mapsto & (V_1 \bigwedge V_5, \bigwedge^2 V) \end{array}$$

that embeds $F_3(G) \cong \mathbb{P}^4$ as a subvariety of the Grassmann variety $G' = G(4,10)$ of 3–planes in $\mathbb{P}^9$. Set $F = g^*\mathcal{S}_4$, and let $\mathcal{Q}_{\mathbb{P}^4}$ be the universal quotient bundle on $\mathbb{P}^4$. As before, one shows that $F = H_1 \otimes H_{5,1} = \mathcal{Q}_{\mathbb{P}^4}(-1)$. Pull back the natural map $S^2(\bigwedge^2 V^\vee) \otimes \mathcal{O}_{G'} \rightarrow S^2(\mathcal{S}_4^\vee)$ to obtain a map $\tilde{s} : S^2(\bigwedge^2 V^\vee) \otimes \mathcal{O}_{\mathbb{P}^4} \rightarrow S^2 F^\vee$. The image $\tilde{s}(Q) \in S^2 F^\vee$ corresponds to a symmetric bundle map $f : F \rightarrow F^\vee$. Let

$$D_k(f) = \{p \in \mathbb{P}^4 : \operatorname{corank} f(p) \geq k\}$$

be the kth degeneracy locus of f. If $(p,h) \in F_X$, then $Q \cap \sigma(p)$ contains the 2–plane $\sigma(p,h)$. Hence p is a singular point, and we have a well–defined map $\tau : F_X \rightarrow C = D_2(f)$ that sends (p,h) to p; the map τ is a double covering, ramified over $D_3(f)$. It follows that if Q is general, the degeneracy loci $\Sigma = D_1(f)$ and $C = D_2(f)$ have the expected codimension; using the formulas in [HT], we find that $\deg \Sigma = 6$ and $\deg C = 40$. I did not verify that $D_3(f) = \emptyset$; if this locus is empty, then σ is an unramified covering and the Riemann–Hurwitz formula shows that the genus of C is 81, as claimed by B. Segre.

3 Infinitesimal Abel–Jacobi map

To study the infinitesimal Abel–Jacobi mapping associated to the family of σ–planes on a general quadratic line complex $X \subset G(2,5)$, we need information on the normal bundle $N_{L,X}$ of a σ–plane $L \subset X$. The normal bundle $N_{L,G}$ has been determined in [P]; we introduce some notation, and then recall the general result.

Let $G(r+1,V)$ be the Grassmann variety of r–planes in $\mathbb{P}(V)$, where V is a complex vector space of dimension $n+1$. We write L_x for the r–plane corresponding to a point $x \in G(r+1,V)$. Let $h \subset \mathbb{P}(V)$ be a hyperplane and $p \in h$ a point. Consider the following types of Schubert cycles:

$$\begin{aligned} Z_1 &= \sigma(h) = \{x \in G : L_x \subset h\} \cong G(r+1,n) \\ Z_2 &= \sigma(p) = \{x \in G : p \in L_x\} \cong G(r,n) \\ Z_3 &= \sigma(p,h) = \{x \in G : p \in L_x \subset h\} \cong G(r,n-1). \end{aligned}$$

Let

$$0 \to \mathcal{S} \to V \otimes \mathcal{O}_G \to \mathcal{Q} \to 0 \tag{4}$$

be the tautological exact sequence on $G(r+1,n+1)$, and let $\mathcal{S}_i$ (resp. $\mathcal{Q}_i$) be the universal subbundle (resp. quotient bundle) on the Grassmann variety Z_i $(i = 1,2,3)$.

Proposition 3.1 *(cf. [P, Prop. 2.4]) The normal bundle of Z_3 in $G = G(r+1,n+1)$ is*

$$N_{Z_3,G} \cong \mathcal{S}_3^\vee \bigoplus \mathcal{Q}_3 \bigoplus \mathcal{O}_{Z_3}.$$

Proof: By comparison of the tautological exact sequence on Z_1 and the restriction of (4) to Z_1 we find an exact sequence

$$0 \to \mathcal{Q}_1 \to \mathcal{Q}_{Z_1} \to \mathcal{O}_{Z_1} \to 0$$

that splits, as $H^1(Z_1, \mathcal{Q}_1) = 0$ by the Bott vanishing theorem. Hence $\mathcal{Q}|_{Z_1} = \mathcal{Q}_1 \bigoplus \mathcal{O}_{Z_1}$. By duality it follows that $\mathcal{S}|_{Z_2} = \mathcal{S}_2 \bigoplus \mathcal{O}_{Z_2}$. The other restrictions are $\mathcal{S}|_{Z_1} = \mathcal{S}_1$, $\mathcal{Q}|_{Z_2} = \mathcal{Q}_2$. As Z_3 is a Scubert cycle of type Z_1 inside Z_2, we obtain

$$\mathcal{S}|_{Z_3} = \mathcal{S}_3 \bigoplus \mathcal{O}_{Z_3}, \ \mathcal{Q}|_{Z_3} = \mathcal{Q}_3 \bigoplus \mathcal{O}_{Z_3}.$$

Hence

$$\begin{aligned} T_G|_{Z_3} &= (\mathcal{S}^\vee \otimes \mathcal{Q})|_{Z_3} \\ &= \mathcal{S}_3^\vee \otimes \mathcal{Q}_3 \bigoplus \mathcal{S}_3^\vee \bigoplus \mathcal{Q}_3 \bigoplus \mathcal{O}_{Z_3} \end{aligned}$$

and the result follows. □

Remark 3.2 The bundles $\mathcal{S}^\vee$ and $\mathcal{Q}$ are not ample (unless they have rank one), as their restrictions to curves contained in Z_3 have a quotient line bundle of degree zero; see [P, Props. 2.2 and 2.3].

We return to the Grassmannian $G = G(2,5)$ of lines in $\mathbb{P}^4$.

Corollary 3.3 *Let $L_0 \subset G = G(2,5)$ be a σ–plane, and let $\mathcal{Q}_{L_0}$ be the universal quotient bundle on $L_0 \cong \mathbb{P}^2$. The normal bundle of L_0 in G is*

$$N_{L_0,G} \cong \mathcal{O}_{L_0}(1) \bigoplus \mathcal{Q}_{L_0} \bigoplus \mathcal{O}_{L_0}.$$

Let $X \subset G$ be a general quadratic line complex. In Lemmas 2.3 and 2.6 we saw that the family of σ–planes on X is parametrized by a smooth, irreducible curve F_X of genus 161. Let

$$\Phi_{F_X} : F_X \to J^3(X)$$

be the Abel–Jacobi mapping associated to this family of planes (note that it is only well–defined up to translation). By the universal property of the Jacobian $J(F_X)$ this map factorizes over a map

$$\Phi : J(F_X) \to J^3(X).$$

Let

$$\begin{array}{ccc} I & \xrightarrow{q} & X \\ \downarrow{\scriptstyle p} & & \\ F_X & & \end{array}$$

be the incidence correspondence. The induced map

$$q_* \circ p^* : H_1(F_X, \mathbb{Z}) \to H_5(X, \mathbb{Z})$$

is called the *cylinder homomorphism* associated to the family F_X. It sends a 1–chain $\gamma \subset F_X$ to the 5–chain $\cup_{x\in\gamma} L_x$ swept out on X by the planes L_x, $x \in \gamma$. Under Poincaré duality the cylinder homomorphism corresponds to a homomorphism

$$\psi_{\mathbb{Z}} : H^1(F_X, \mathbb{Z}) \to H^5(X, \mathbb{Z}).$$

Its complexification $\psi_{\mathbb{C}}$ is a morphism of Hodge structures of type $(2,2)$ that induces a map

$$\psi : H^0(\Omega^1_{F_X})^\vee = H^{0,1}(F_X) \to H^{2,3}(X) = H^2(\Omega^3_X)^\vee.$$

Choose a point $0 \in F_X$ and let $L_0 \subset X$ be the corresponding σ–plane. The following result is due to Griffiths and Welters. Note that the adjunction formula shows that $\det(N_{L_0,X}) \cong \mathcal{O}_{L_0}$.

Lemma 3.4.

(i) The transpose of the infinitesimal Abel–Jacobi mapping is the composition of the maps

$$\begin{array}{rcl} H^2(X,\Omega^3_X) & \longrightarrow & H^2(L_0, \Omega^3_X|_{L_0}) \\ H^2(L_0,\Omega^3_X|_{L_0}) & \longrightarrow & H^2(L_0, K_{L_0} \otimes \bigwedge^2 N_{L_0,X}) \\ H^2(L_0, K_{L_0} \otimes \bigwedge^2 N_{L_0,X}) & \xrightarrow{\sim} & H^0(L_0, N_{L_0,X})^\vee \\ H^0(L_0, N_{L_0,X})^\vee & \xrightarrow{\sim} & T^\vee_{F_X,0} \end{array}$$

(ii) The composed map

$$\tau : H^2(X, \Omega^3_X) \to H^2(L_0, K_{L_0} \otimes \textstyle\bigwedge^2 N_{L_0,X})$$

fits into a commutative diagram

$$\begin{array}{ccc} H^1(X, T_X(-1)) & \xrightarrow{\beta} & H^1(L_0, N_{L_0,X}(-1)) \\ \downarrow & & \downarrow \\ H^2(X,\Omega^3_X) & \xrightarrow{\tau} & H^2(L_0, \bigwedge^2 N_{L_0,X}(-3)) \\ \Big\downarrow & & \Big\downarrow \alpha \\ H^2(X, \bigwedge^2 T_G \otimes \mathcal{O}_X(-3)) & \longrightarrow & H^2(L_0, \bigwedge^2 N_{L_0,G}(-3)) \\ \downarrow & & \downarrow \\ \cdots & \longrightarrow & H^2(L_0, N_{L_0,X}(-1)) \\ \downarrow & & \downarrow \\ \cdots & \longrightarrow & 0 \end{array}$$

with exact columns.

Proof: For (i), see [G, Thm. 2.25]. Part (ii) is essentially due to Welters [Wel]: take exterior powers in the two bottom rows of the commutative diagram

$$(*)\qquad \begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & T_X \otimes \mathcal{O}_{L_0} & \to & T_G \otimes \mathcal{O}_{L_0} & \to & N_{X,G} \otimes \mathcal{O}_{L_0} & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & N_{L_0,X} & \to & N_{L_0,G} & \to & N_{X,G} \otimes \mathcal{O}_{L_0} & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

and take the tensor product with $K_X \otimes \mathcal{O}_{L_0}$ to obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \bigwedge^2 T_X \otimes \mathcal{O}_{L_0} & \to & \bigwedge^2 T_G \otimes \mathcal{O}_{L_0} & \to & T_X \otimes N_{X,G} \otimes \mathcal{O}_{L_0} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \bigwedge^2 N_{L_0,X} & \to & \bigwedge^2 N_{L_0,G} & \to & N_{L_0,X} \otimes N_{X,G} & \to & 0. \end{array}$$

The desired commutative diagram is obtained from the associated long exact sequences in cohomology by composing with the map on cohomology groups induced by the restriction $\mathcal{O}_X \to \mathcal{O}_{L_0}$. □

Lemma 3.5.

(i) $\ker \alpha \neq 0$.

(ii) β *is surjective.*

Proof: (i): The Hilbert scheme Hilb_X^P that parametrizes 2–planes in X is the union of F_X and a finite number of points (corresponding to the ρ–planes contained in X). Hence the tangent space at 0 to F_X is isomorphic to $H^0(L_0, N_{L_0,X})$. As

$$h^2(L_0, \bigwedge^2 N_{L_0,X}(-3)) = h^0(L_0, N_{L_0,X}) = 1$$

by Serre duality, Lemma 3.4 shows that

$$\ker \alpha \neq 0 \iff H^2(L_0, \bigwedge^2 N_{L_0,G}(-3)) = H^2(L_0, N_{L_0,X}(-1)).$$

We shall show that both cohomology groups vanish. By Corollary 3.3 we have

$$\bigwedge^2 N^\vee_{L_0,G} \cong \bigwedge^2(\mathcal{Q}^\vee_{L_0} \oplus \mathcal{O}_{L_0}(-1) \oplus \mathcal{O}_{L_0}) \cong \bigoplus^2 \mathcal{O}_{L_0}(-1) \oplus \Omega^1_{L_0} \oplus \mathcal{Q}^\vee_{L_0},$$

hence

$$H^2(L_0, \bigwedge^2 N_{L_0,G}(-3)) \cong H^0(L_0, \bigwedge^2 N^\vee_{L_0,G})^\vee = 0.$$

By Lemma 3.4 (ii) it follows that $H^2(L_0, N_{L_0,X}(-1) = 0$.

For part (ii), we note that the commutative diagram (*) of Lemma 3.4 induces a commutative diagram

$$\begin{array}{ccc} H^0(X, \mathcal{O}_X(1)) & \xrightarrow{\gamma_1} & H^0(L_0, \mathcal{O}_{L_0}(1)) \\ \downarrow & & \downarrow \gamma_2 \\ H^1(X, T_X(-1)) & \xrightarrow{\beta} & H^1(L_0, N_{L_0,X}(-1)). \end{array}$$

The map γ_1 is surjective, since X and L_0 are projectively normal in $\mathbb{P}^9$. Corollary 3.3 shows that

$$H^1(L_0, N_{L_0,G}(-1)) = H^1(L_0, \mathcal{Q}_{L_0}(-1) \bigoplus \mathcal{O}_{L_0} \bigoplus \mathcal{O}_{L_0}(-1)) = 0,$$

hence the map γ_2 is also surjective. Thus β is surjective. □

Corollary 3.6. *The map $\Phi : J(F_X) \to J^3(X)$ is nontrivial.*

Proof: A diagram chase shows that $\tau = \Phi_*$ is nontrivial; hence Φ is nontrivial.

□

Let $\{X_t\}_{t \in \mathbb{P}^1}$ be a Lefschetz pencil in $\mathbb{P}H^0(\mathbb{P}^9, \mathcal{O}_{\mathbb{P}^9}(2))$ with $X_0 = X$. Let

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{q} & \mathcal{X} \\ \Big\downarrow p & & \\ \mathcal{F} & & \end{array}$$

be the relative incindence correspondence. Let $U' \subset \mathbb{P}^1$ (resp. $U'' \subset \mathbb{P}^1$) be the subset over which $\mathcal{X}$ (resp. $\mathcal{F}$) is smooth. Set $U = U' \cap U''$.

Theorem 3.7. *If $X \subset G(2,5)$ is a general quadratic line complex, the map $\Phi : J(F_X) \to J^3(X)$ is surjective.*

Proof: Since the cylinder homomorphism is equivariant with respect to the action of $\pi_1(U,0)$ ans the fundamental group $\pi_1(U',0)$ acts transitively on $H^5(X,\mathbb{Q})$ (cf. [V, Lecture 4]), the surjectivity of ψ and Φ follows from Corollary 3.6, because the images of $\pi_1(U,0)$ and $\pi_1(U',0)$ in Aut $H^5(X,\mathbb{Z})$ coincide. □

Corollary 3.8 *If $X_subsetG(2,5)$ is a general quadratic line complex, the generalized Hodge conjecture* GHC$(X,5,2)$ *holds.*

Proof: As the map $\Phi_{F_X} : F_X \to J^3(X)$ factors through the Abel–Jacobi map $\psi_X : \mathrm{CH}^3_{\mathrm{alg}}(X) \to J^3(X)$, Theorem 3.7 shows that $J^3_{\mathrm{alg}}(X) = J^3(X) = J^3_{\max}(X)$. Then apply [Mu, Lemma 4.3]. □

Remark 3.9 (i) The variety X is rational. This can be proved by projecting from one of the finitely many $_rho$–planes contained in X; see e.g. [Rot, p. 96] or [Sem, 6.3]. A different proof is obtained by projecting from a σ–plane contained in X; this maps X birationally onto an irreducible quadric in $\mathbb{P}^6$.

(ii) For very general complete intersections of sufficiently high multidegree, the image of the Abel–Jacobi map is much smaller. The following theorem is a special case of a result for complete intersections in Grassmann varieties proved in [Na]:

Theorem 3.10 *Let $X = V(d_0,\ldots,d_r)$ ($d_0 \geq \ldots \geq d_r$, $r \leq 2$) be a smooth complete intersection of dimension $2m-1$ ($2 \leq m \leq 3$) in $G = G(2,5)$; let $i : X \to G$ be the inclusion map. If X is very general, then the image of the rational Deligne cycle class map*

$$\mathrm{cl}_{\mathcal{D},X} : \mathrm{CH}^m(X) \otimes \mathbb{Q} \to H^{2m}_{\mathcal{D}}(X,\mathbb{Q}(m))$$

coincides with the image of the composed map

$$i^* \circ \mathrm{cl}_{\mathcal{D},G} : \mathrm{CH}^m(G) \otimes \mathbb{Q} \to H^{2m}_{\mathcal{D}}(X,\mathbb{Q}(m)),$$

except possibly if

(i) *(m = 3)* $X = V(2)$;

(ii) *(m = 2)* $X = V(d, 1, 1)$, $d \geq 1$ *or* $X = V(d, 2, 1)$, $d \geq 2$.

The assertion about the image of the rational Deligne cycle class map in Theorem 3.10 implies that the image of the Abel–Jacobi map

$$\psi_X : \mathrm{CH}^m_{\mathrm{hom}}(X) \to J^m(X)$$

is, up to torsion, determined by the group $\mathrm{Hdg}^m_{\mathrm{pr}}(G)$ of primitive Hodge classes on $G(2, 5)$. More precisely, we can show that up to torsion every normal function defind over a finite étale covering of the moduli space of complete intersections of multidegree $(d_0, \ldots, d_r)$ $(r = 6 - 2m)$ is induced by an algebraic cycle $Z \in \mathrm{CH}^m(G)$ whose cycle class belongs to $\mathrm{Hdg}^m_{\mathrm{pr}}(G)$; for $m = 3$ this means that such a normal function is a torsion section of the fiber space of intermediate Jacobians, as $\mathrm{Hdg}^3_{\mathrm{pr}}(G) = 0$. Theorem 3.7 shows that Theorem 3.10 is sharp in case (i) $(m = 3)$; about case (ii) I do not know, except for low values of d that give rise to Fano or Calabi–Yau threefolds.

References

[AK] A. Altman and S. Kleiman, *Foundations of the theory of Fano schemes*, Comp. Math. **34** (1977), 3–47.

[Bar 1] F. Bardelli, A footnote to a paper by A. Grothendieck, Rend. Semin. Mat. Fis. Milano **57** (1987), 109–124.

[Ba 2] F. Bardelli, *On Grothendieck's generalized Hodge conjecture for a family of threefolds with trivial canonical bundle*, J. reine und angew. Math. **422** (1993), 165–200.

[Bott] R. Bott, *Homogeneous vector bundles*, Ann. of Math. **66** (1957), 203–248.

[C] A. Collino, *The Abel–Jacobi isomorphism for cubic fivefolds*, Pacific J. Math. **122** (1986), 43–56.

[Don] R. Donagi, *On the geometry of Grassmannians*, Duke Math. J. **44** (1977), 795–837.

[FH] W. Fulton and J. Harris, *Representation theory. A first course*, GTM **129**, Springer–Verlag (1991).

[G] P. Griffiths, *Periods of integrals on algebraic manifolds II*, Am. J. Math. **90** (1968), 805–864.

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley (1978).

[HAG] R. Hartshorne, *Algebraic Geometry*, GTM **52**, Springer–Verlag (1977).

[Hu] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, GTM **9**, Springer–Verlag (1972).

[HT] J. Harris and L.W. Tu, *On symmetric and skew–symmetric determinantal varieties*, Topology **23** (1984), 71–84.

[JPW] T. Józefiak, P. Pragacz and J. Weyman, *Resolutions of determinantal varieties*, in: Young tableaux and Schur functors in algebra and geometry, Torún, Poland 1980, Astérisque **87–88** (1981), 109–189.

[Ma] D.G. Markusevic, *Numerical invariants of families of lines on some Fano varieties*, Math. USSR Sbornik **44** (1983), 239–260.

[Mi] Y. Miyaoka, On the Kodaira dimension of minimal threefolds, Math. Ann. **281** (1988), 325–332.

[Mu] J.P. Murre, Abel–Jacobi equivalence versus incidence equivalence for algebraic cycles of comdimension two, Topology **24** (1985), 361–367.

[Na] J. Nagel, The image of the Abel–Jacobi map for complete intersections, Thesis, University of Leiden (1997).

[P] A. Papantanopoulou, *Curves in Grassmann varieties*, Nagoya Math. J. **66** (1977), 121–137.

[Re] M. Reid, Thesis, Cambridge (1972).

[Ros] M. Rossi, Hodge theory on some invariant threefolds of even degree, Indag. Math. **8** (1997), 267–279.

[Rot] L. Roth, *Some properties of Grassmannians*, Rend. di Mat. **5** (1951), 96–114.

[Seg] B. Segre, *Studio dei complessi quadratici di rette di* S_4, Atti Ist. Veneto **88** (1929), 595–649.

[Sem] J. Semple, On quadric representations of the lines of four–dimensional space, Proc. London Math. Soc. **30** (1930), 500–512.

[Sh] T. Shioda, What is known about the Hodge conjecture ?, in: Advanced Studies in Pure Math. **1** (1983), 55–68.

[St] J.H.M. Steenbrink, Some remarks on the Hodge conjecture, in: Hodge theory, San Cugat, Lecture Notes in Mathematics **1246** (1987), 165–175.

[SR] J.G. Semple and L. Roth, *Introduction to Algebraic Geometry*, Clarendon Press, Oxford (1985) (paperback reprint).

[T] A.N. Tyurin, *On intersections of quadrics*, Russian Math. Surveys **30** (1975), 51–105.

[V] C. Voisin, lectures at Sophia–Antipolis, 1991.

[Weh] J. Wehler, *Deformation of varieties defined by sections in homogeneous vector bundles*, Math. Ann. **268** (1984), 519–532.

[Wel] G. Welters, *Abel–Jacobi isogenies for certain types of Fano threefolds*, Math. Center Tract **141**, Amsterdam (1981).