Effective bounds for Hodge–theoretic connectivity

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Abstract

We prove an effective version of Nori's connectivity theorem using Koszul cohomology computations. We apply this result to study the cycle class, Abel–Jacobi and regulator maps and the nonvanishing of certain Griffiths groups for complete intersections in projective spaces, abelian varieties and quadrics.

1 Introduction

Let X be a smooth projective variety of dimension n defined over $\mathbb{C}$. To study the nature of the Chow group $\mathrm{CH}^p(X)$ of algebraic cycles of codimension p on X one can use the cycle class map

$$\mathrm{cl}^p_X : \mathrm{CH}^p(X) \to H^{2p}(X, \mathbb{Z})$$

and the Abel–Jacobi map

$$\psi^p_X : \mathrm{CH}^p_{\mathrm{hom}}(X) \to J^p(X).$$

For $p \geq 2$ little is known about the images of these maps. If X is a very general smooth hypersurface of degree d in $\mathbb{P}^{n+1}$, the images of these maps (in the interesting cases $2p = \dim X$ and $2p - 1 = \dim X$) are described by the following two theorems:

Theorem 1. (Noether–Lefschetz) [D], [Sh, Thm. 2.1] *If $X = V(d) \subset \mathbb{P}^{2m+1}$ is a very general smooth hypersurface of degree $d \geq 2 + 2/m$ and dimension $2m$ ($m \geq 1$), the image of cl^m_X is isomorphic to $\mathbb{Z}$ and is generated by the class of a hyperplane section.*

Theorem 2. (Green–Voisin) [G4] *If $X = V(d) \subset \mathbb{P}^{2m}$ is a very general smooth hypersurface of degree $d \geq 2 + 4/(m-1)$ and dimension $2m - 1$ ($m \geq 2$), the image of ψ^m_X is contained in the torsion points of $J^m(X)$.*

M.V. Nori [No] has obtained a remarkable generalisation of these results. His main result is a connectivity theorem for the universal family $X_S \subset Y_S$ of complete intersections of multidegree $(d_0, \dots, d_r)$ and dimension n in a smooth projective variety Y: for every nonnegative integer $c \leq n$ there exists a natural number $N(c)$ such that for every smooth morphism $T \to S$ we have $H^{n+k}(Y_T, X_T; \mathbb{Q}) = 0$ for all $k \leq c$ if $\min(d_0, \dots, d_r) \geq N(c)$.

For $Y = \mathbb{P}^{n+1}$, the asymptotic versions (i.e., without explicit degree bounds) of the theorems of Noether–Lefschetz and Green–Voisin follow from Nori's connectivity theorem: they correspond to the cases $c = 1$ and $c = 2$ (see Section 4 for details). Effective versions of Nori's theorem have been worked out in [Pa], [BM] and [R]. The degree bounds obtained in these papers do not suffice to recover Theorems 1 and 2 from Nori's connectivity theorem: the most precise result, due to Paranjape [Pa], yields the bound $d \geq 2m + 2$ in both cases.

In this paper we present a different proof of Nori's theorem, based on the original method of Green and Voisin. This method leads to more precise degree bounds; in particular, we find the (optimal) degree bounds of Theorems 1 and 2. Our condition on the base change is stronger than the one in Nori's theorem: we assume that the induced family X_T is smooth over T, i.e., the smooth morphism $T \to S$ factors through a (necessarily smooth) morphism from T to $U = S \setminus \Delta$, the complement of the discriminant locus. This condition suffices to treat the known geometric applications of Nori's theorem.

In Section 2 we recall the basic ideas of Nori's proof and some technical results on spectral sequences. In Section 3 we prove an effective version of Nori's connectivity theorem using Koszul cohomology computations; our method is partly based on unpublished manuscripts of Green and Müller–Stach [GM2]. (I was told that there exists also related unpublished work of Nori.) In the case $Y = \mathbb{P}^N$, similar degree bounds have been obtained by M. Asakura and S. Saito [AS]. In Section 4 we apply Nori's connectivity theorem to complete intersections inside projective spaces, quadrics and abelian varieties to obtain effective versions of results concerning the image of the cycle class, Abel–Jacobi and regulator maps and the nonvanishing of certain Griffiths groups. The results in this paper have been announced in [Na2].

Notation and conventions. We work over the field of complex numbers. Unless stated otherwise, cohomology is taken with coefficients in $\mathbb{C}$. For an abelian group G we write $G_{\mathbb{Q}} = G \otimes \mathbb{Q}$. If $f : C^\bullet \to D^\bullet$ is a map of complexes, the mapping cone $C^\bullet(f)$ fits into a short exact sequence

$$0 \to D^\bullet[-1] \to C^\bullet(f) \to C^\bullet \to 0;$$

this differs from the usual definition by a shift of one.

2 Review of Nori's results

We recall the setup for Nori's theorem and the main ideas and technical ingredients of his proof. Let $(Y, \mathcal{O}_Y(1))$ be a smooth polarised variety defined over $\mathbb{C}$, and let X be a smooth complete intersection of dimension n in Y, defined by a global section of the vector bundle $E = \mathcal{O}_Y(d_0) \oplus \ldots \oplus \mathcal{O}_Y(d_r)$. Set $S = \mathbb{P}H^0(Y, E)$, let $X_S \subset Y_S = Y \times S$ be the universal family of complete intersections of multidegree $(d_0, \ldots, d_r)$ in Y and let $h : T \to S$ be a smooth morphism. By base change we obtain a family $X_T = X \times_S T$ inside the trivial family $Y_T = Y \times T$. Let $i : X_T \to Y_T$ be the inclusion map, and let $p_T : Y_T \to T$ and $f = p_T \circ i : X_T \to T$ be the projections to T.

Theorem 2.1. **(Nori)** [No, Thm. 4] *For every natural number $c \leq n$ there exists a natural number $N = N(Y, \mathcal{O}_Y(1), c)$ such that for every smooth morphism $h : T \to S$ we have $H^{n+k}(Y_T, X_T; \mathbb{Q}) = 0$ for all $k \leq c$ if $\min(d_0, \ldots, d_r) \geq N$.*

Remark 2.2.

(i) By the Lefschetz hyperplane theorem, the restriction map $R^q(p_T)_*\mathbb{Z} \to R^q f_*\mathbb{Z}$ is an isomorphism if $q < n$ and is injective if $q = n$. Using the Leray spectral sequence we find that $H^k(Y_T, X_T; \mathbb{Z}) = 0$ for all $k \leq n$. For $k > n$ there are examples showing that $H^{n+k}(Y_T, X_T; \mathbb{Z})$ may be nonzero even if $\min(d_0, \ldots, d_r) \gg 0$.

(ii) There is some freedom in the choice of the base S: the assertion of Theorem 2.1 remains valid if we choose $S = \prod_{i=0}^r \mathbb{P}H^0(Y, \mathcal{O}_Y(d_i))$ (as in Nori's paper), $S = H^0(Y, E)$ or $S = \mathbb{P}H^0(Y, E) \setminus \Delta$, where Δ denotes the discriminant locus; see [No, Remark 3.3], [G5, Lecture 8] or Lemma 2.6.

The proof of Theorem 2.1 uses mixed Hodge theory. Let $Y_T \subset \overline{Y}_T$ and $X_T \subset \overline{X}_T$ be good compactifications with boundary divisors $D_T = \overline{Y}_T \setminus Y_T$ and $D'_T = D_T \cap \overline{X}_T$, and let $\alpha : \Omega^\bullet_{Y_T} \longrightarrow i_*\Omega^\bullet_{X_T}$ and $\beta : \Omega^\bullet_{\overline{Y}_T}(\log D_T) \longrightarrow i_*\Omega^\bullet_{\overline{X}_T}(\log D'_T)$ be the restriction maps. The mapping cones $C^\bullet(\alpha)$ and $C^\bullet(\beta)$ fit into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & i_*\Omega^\bullet_{\overline{X}_T}(\log D'_T)[-1] & \to & C^\bullet(\beta) & \to & \Omega^\bullet_{\overline{Y}_T}(\log D_T) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & i_*\Omega^\bullet_{X_T}[-1] & \to & C^\bullet(\alpha) & \to & \Omega^\bullet_{Y_T} & \to & 0. \end{array}$$

The Hodge filtration on $H^{n+k}(Y_T, X_T) \cong H_c^{n+k}(Y_T \setminus X_T)$ is induced by the filtration bête on $C^\bullet(\beta)$ [DD, Lemme 2.2]:

$$F^p H^{n+k}(Y_T, X_T) = \mathrm{im}(\mathbb{H}^{n+k}(\sigma_{\geq p} C^\bullet(\beta)) \to \mathbb{H}^{n+k}(C^\bullet(\beta))).$$

Nori deduced Theorem 2.1 from the following result:

Theorem 2.3. **(Nori)** [No, Thm. 3] *For every natural number c there exists a natural number $N = N(Y, \mathcal{O}_Y(1), c)$ such that for every smooth morphism $h : T \to S$ we have $F^k H^{n+k}(Y_T, X_T) = 0$ for all $k \leq c$ if $\min(d_0, \ldots, d_r) \geq N$.*

The cohomology group $H^{n+k}(Y_T, X_T)$ carries an increasing weight filtration $W_\bullet$ that is strictly compatible with the weight filtrations on $H^{n+k}(Y_T)$ and $H^{n+k}(X_T)$. As Y_T and X_T are smooth quasi–projective varieties, it follows that

$$\mathrm{Gr}_i^W H^{n+k}(Y_T, X_T) = 0 \quad \text{if } i < n + k - 1. \tag{1}$$

To see how Theorem 2.1 follows from Theorem 2.3, assume that one of the Hodge numbers $h^{p,q}$ of $H^{n+k}(Y_T, X_T)$ is nonzero. By Theorem 2.3 we have $p \leq k-1$ and (by symmetry) $q \leq k-1$. Hence if $k \leq n$ then $p+q \leq 2k-2 < n + k - 1$, but this contradicts (1).

To prove Theorem 2.3, Nori introduced a filtration $G^\bullet$ on $H^{n+k}(Y_T, X_T)$ that is coarser than the Hodge filtration $F^\bullet$ but better suited for computations. The complex $C^\bullet(\alpha)$ is quasi–isomorphic to the complex $\Omega^\bullet_{Y_T, X_T} = \ker(\Omega^\bullet_{Y_T} \to i_* \Omega^\bullet_{X_T})$. Hence by Grothendieck's algebraic De Rham theorem [Gr] and the five lemma we have

$$\mathbb{H}^{n+k}(C^\bullet(\alpha)) \cong H^{n+k}(Y_T, X_T).$$

Define

$$G^p H^{n+k}(Y_T, X_T) = \mathrm{im}(\mathbb{H}^{n+k}(\sigma_{\geq p} C^\bullet(\alpha)) \to \mathbb{H}^{n+k}(C^\bullet(\alpha))).$$

The commutative diagram that relates $C^\bullet(\alpha)$ and $C^\bullet(\beta)$ shows that

$$F^p H^{n+k}(Y_T, X_T) \subset G^p H^{n+k}(Y_T, X_T).$$

To prove Theorem 2.1 it thus suffices to show that $G^k H^{n+k}(Y_T, X_T) = 0$ for all $k \leq c$. A closer look at Nori's weight argument reveals that it suffices to show that for all $k \leq c$ we have $G^{b_k} H^{n+k}(Y_T, X_T) = 0$ for some natural number $b_k \leq [\frac{n+k}{2}]$. Nori's choice is $b_k = k$ $(k = 1, \ldots, c)$. Our choice is

$$b_k = \left[\frac{n-c}{2}\right] + k \tag{2}$$

In [Pa] K. Paranjape proved an effective version of Theorem 2.1 using Castelnuovo–Mumford regularity. Let m_j be the Castelnuovo–Mumford regularity of the vector bundle Ω^j_Y, i.e.,

$$m_j = \min\{k \in \mathbb{Z} | H^i(Y, \Omega^j_Y(k-i)) = 0 \text{ for all } i > 0\}.$$

Following Paranjape we define $m_Y = \max\{m_i - i - 1 : 0 \le i \le \dim\, Y\} \in \mathbb{Z}_{\ge 0}$.

Theorem 2.4. **(Paranjape)** [Pa, (2.3)] *With the notation of Theorem 2.1, we have*

$$N(Y, \mathcal{O}_Y(1), c) \le m_Y + n + c + 1.$$

Definition 2.5. *Let $g : T \to U = \mathbb{P}H^0(Y, E) \setminus \Delta$ be a smooth morphism, and let c be a natural number. We say that Nori's condition (N_c) holds for the pair (Y_T, X_T) if for all $k \le c$*

$$R^a(p_T)_*\Omega^b_{Y_T,X_T} = 0 \quad \text{for all pairs } (a,b) \text{ with } a+b \le n+k, \quad b \ge b_k.$$

Lemma 2.6. *Let $g : T \to U$ be a smooth morphism. Then*

(i) If (N_c) holds for (Y_U, X_U), then (N_c) holds for (Y_T, X_T).

(ii) If g is surjective and (N_c) holds for (Y_T, X_T), then (N_c) holds for (Y_U, X_U).

Proof: See [No, Lemma 2.2]. Note that the replacement of Nori's choice $b_k = k$ by our choice of the numbers b_k does not affect the proof, as we have $b_{k-p} = b_k - p$ for all $p \le k$. □

Lemma 2.7. *If $f : X_T \to T$ is smooth and if condition (N_c) holds for (Y_T, X_T) then $H^{n+k}(Y_T, X_T, \mathbb{Q}) = 0$ for all $k \le c$.*

Proof: It suffices to show that $G^{b_k}H^{n+k}(Y_T, X_T) = 0$ for all $k \le c$, where the numbers $b_1, \dots, b_c$ are chosen as in (2). Consider the Grothendieck spectral sequence of composite functors

$$E_2^{p,q} = H^p(T, \mathbb{R}^q(p_T)_*\Omega^\bullet_{Y_T,X_T}) \Rightarrow \mathbb{H}^{p+q}(\Omega^\bullet_{Y_T,X_T}) \cong H^{p+q}(Y_T, X_T).$$

Using this spectral sequence for the filtered complex $\sigma_{\ge b_k}\Omega^\bullet_{Y_T,X_T}$, we find that $G^{b_k}H^{n+k}(Y_T, X_T) = 0$ if $\mathbb{R}^q(p_T)_*\sigma_{\ge b_k}\Omega^\bullet_{Y_T,X_T} = 0$ for all $q \le n+k$. Using the spectral sequence $E_1^{a,b} \Rightarrow \mathbb{R}^{a+b}(p_T)_*\sigma_{\ge b_k}\Omega^\bullet_{Y_T,X_T}$ with $E_1^{a,b} = R^a(p_T)_*\sigma_{\ge b_k}\Omega^b_{Y_T,X_T}$ we find that $\mathbb{R}^q(p_T)_*\sigma_{\ge b_k}\Omega^\bullet_{Y_T,X_T} = 0$ for all $q \le n+k$ and all $k \le c$ if condition (N_c) holds. □

Let $i_t : Y \to Y_T$ be the inclusion map defined by $i_t(y) = (y, t)$.

Lemma 2.8. *If $f : X_T \to T$ is smooth, there exists for every $t \in T$ a spectral sequence*

$$E_1^{p,q}(b) = \Omega^p_{T,t} \otimes H^{p+q}(Y, \Omega^{b-p}_{Y,X_t}) \Rightarrow H^{p+q}(Y, i_t^* \Omega^b_{Y_T,X_T}).$$

Proof: As $f : X_T \to T$ is smooth, we have an exact sequence

$$0 \to f^*\Omega^1_T \to \Omega^1_{X_T} \to \Omega^1_{X_T/T} \to 0$$

that induces an increasing filtration $L^\bullet$, the Leray filtration, on $\Omega^\bullet_{X_T}$:

$$L^p\Omega^\bullet_{X_T} = \mathrm{im}(f^*\Omega^p_T \otimes \Omega^\bullet_{X_T}[-p] \to \Omega^\bullet_{X_T}).$$

The split exact sequence

$$0 \to f^*\Omega^1_T \to \Omega^1_{Y_T} \to p_T^*\Omega^1_T \to 0$$

induces a filtration $L^\bullet$ on $\Omega^\bullet_{Y_T}$. Define

$$\Omega^\bullet_{(Y_T,X_T)/T} = \ker(\Omega^\bullet_{Y_T/T} \to i_*\Omega^\bullet_{X_T/T}).$$

We have an induced filtration

$$L^p\Omega^\bullet_{Y_T,X_T} = \ker(L^p\Omega^\bullet_{Y_T} \to L^p i_*\Omega^\bullet_{X_T})$$

with graded pieces

$$\mathrm{Gr}^p_L\, \Omega^\bullet_{Y_T,X_T} \cong f^*\Omega^p_T \otimes \Omega^\bullet_{(Y_T,X_T)/T}[-p].$$

The induced filtration on $i_t^*\Omega^\bullet_{Y_T,X_T}$ defines a spectral sequence

$$E_1^{p,q}(b) = H^{p+q}(Y, \mathrm{Gr}^p_L\, i_t^*\Omega^b_{Y_T,X_T}) \Rightarrow H^{p+q}(Y, i_t^*\Omega^b_{Y_T,X_T})$$

with $E_1^{p,q}(b) \cong \Omega^p_{T,t} \otimes H^{p+q}(Y, \Omega^{b-p}_{Y,X_t})$.

$\square$

Remark 2.9. In a similar way one can construct spectral sequences

$$\begin{aligned} E_1^{p,q}(X_t, b) &= \Omega^p_{T,t} \otimes H^{p+q}(X_t, \Omega^{b-p}_{X_t}) \Rightarrow H^{p+q}(X_t, \Omega^b_{X_T} \otimes \mathcal{O}_{X_t}) \\ E_1^{p,q}(Y, b) &= \Omega^p_{T,t} \otimes H^{p+q}(Y, \Omega^{b-p}_Y) \Rightarrow H^{p+q}(Y, \Omega^b_{Y_T} \otimes \mathcal{O}_Y). \end{aligned}$$

Let $j : X_t \to Y$ be the inclusion map. Define

$$\begin{aligned} E_1^{\bullet,q}(X_t, b)_{\mathrm{var}} &= \mathrm{coker}\,(j^* : E_1^{\bullet,q}(Y, b) \to E_1^{\bullet,q}(X_t, b)) \\ E_1^{\bullet,q}(Y, b)_0 &= \ker(j^* : E_1^{\bullet,q}(Y, b) \to E_1^{\bullet,q}(X_t, b)). \end{aligned}$$

The map d_1 in the spectral sequence $E_1^{\bullet,q}(X_t, b)$ can be identified with the differential of the period map and is given by cup product with the Kodaira–Spencer class, followed by contraction; cf. [G5, Lecture 3]. The complex $E_1^{\bullet,q}(b)$ fits into a short exact sequence of complexes

$$0 \to E_1^{\bullet,q-1}(X_t, b)_{\mathrm{var}} \to E_1^{\bullet,q}(b) \to E_1^{\bullet,q}(Y, b)_0 \to 0.$$

By the Lefschetz hyperplane theorem we have $E_1^{p,q}(b) = 0 \quad$ if $b + q \le n$. As we shall see in Section 3 the term $E_1^{p,q}(b)$ (or rather its dual) can be expressed using Jacobi rings if $b + q = n + 1$ and if the degrees d_i are sufficiently large. If $b+q > n+1$, say $b+q = n+1+e$, it follows from the Lefschetz hyperplane theorem and the hard Lefschetz theorem that we have an isomorphism

$$E_1^{p,q}(b) \cong E_1^{p,q}(b)_0 \cong \Omega^p_{T,t} \otimes H^{b-p-e,p+q-e}_{\mathrm{pr}}(Y).$$

The only nonzero differentials in the spectral sequence $E_r^{p,q}(b)$ are the maps $d_r : E_r^{p-r,q+r-1}(b) \to E_r^{p,q}(b)$ with $b + q = n + 1$. If Y has no primitive cohomology, the spectral sequence $E_r^{p,q}(b)$ degenerates at E_2. In general we have $E_\infty^{p,q}(b) = E_{k+1}^{p,q}(b)$ if $p + q + b \le n + k$. If $E_\infty^{p,q}(b) = 0$ there exists an increasing filtration on $E_2^{p,q}(b)$ $(b + q = n + 1)$ whose graded pieces are controlled by the primitive cohomology of Y. A proof of this statement was announced in [Mul1, Thm. 1.7] but a proof never appeared.

Lemma 2.10. *Let $\Delta' \subset H^0(Y, E)$ be the discriminant locus. If $E_\infty^{p,q}(b) = 0$ for all $t \in U' = H^0(Y, E) \setminus \Delta'$, for all $k \le c$ and for all (p, q, b) such that $p + q + b \le n + k$, $b \ge b_k$, then for every smooth morphism $g : T \to U = \mathbb{P}H^0(Y, E) \setminus \Delta$ we have $H^{n+k}(Y_T, X_T) = 0$ for all $k \le c$.*

Proof: If $H^{p+q}(Y, i_t^*\Omega^b_{Y_T,X_T}) = 0$ for all $t \in T$ then $R^{p+q}(p_T)_*\Omega^b_{Y_T,X_T} = 0$ by semicontinuity; cf. [Mum, Cor. 2, p. 50]. Hence, if the conditions of the Lemma are satisfied then Nori's condition (N_c) holds for the pair $(Y_{U'}, X_{U'})$. The composition of the inclusion map $U' \to H^0(Y, E)\setminus\{0\}$ and the projection to $\mathbb{P}H^0(Y, E)$ is a smooth morphism $h : U' \to \mathbb{P}H^0(Y, E)$ whose image is U. By Lemma 2.6 (ii) it follows that condition (N_c) holds for (Y_U, X_U), hence condition (N_c) holds for (Y_T, X_T) by Lemma 2.6 (i) and the assertion follows by Lemma 2.7. $\square$

Remark 2.11. We have stated Nori's condition (N_c) and Lemma 2.6 using the pair (Y_U, X_U), $U = \mathbb{P}H^0(Y, E)\setminus\Delta$. The conclusion of Lemma 2.6 remains valid if we replace the pair (Y_U, X_U) by a pair (A, B) of smooth varieties over $\mathbb{C}$ such that $i : B \to A$ is a closed immersion and A admits a morphism $p : A \to U$; see [No, Section 2]. If one assumes in addition that the morphism p is smooth, all the results in this section that have been stated for the pair (Y_U, X_U) remain valid for the pair (A, B).

Remark 2.12. In his paper, Nori works with the projection map $p_Y \circ i : X_T \to Y$, which is a smooth morphism for every base change $T \to S$. The use of the other projection map $f = p_T \circ i : X_T \to T$ allows us to make the connection with the the theory of infinitesimal variations of Hodge structure and the work of Green and Voisin, at the cost of a slightly stronger assumption on the base change (it has to factor through the complement U of the discriminant locus). See also [No, Remark 3.10].

In the case $c = 2$ and $Y = \mathbb{P}^{n+r+1}$ our method for the proof of Nori's theorem is essentially the method of Green and Voisin, phrased in terms of the cohomology of the universal family. To see this, take $n = 2m - 1$ and note that by the Lefschetz hyperplane theorem we have $E_1^{p,q}(b) = 0$ for all $b + q \le 2m - 1$. By Lemma 2.10 we have $H^{n+k}(Y_T, X_T) = 0$ for all $k \le 2$ if

(i) $E_\infty^{0,2m-b}(b) = E_2^{0,2m-b}(b) = 0$ for all $b \ge m - 1$

(ii) $E_\infty^{1,2m-b}(b) = E_2^{1,2m-b}(b) = 0$ for all $b \ge m$

(iii) $E_\infty^{0,2m+1-b}(b) = E_3^{0,2m+1-b}(b) = 0$ for all $b \ge m$.

Set $\mathbb{V} = R^{2m-1} f_* \mathbb{Z}$, $\mathcal{V} = \mathbb{V} \otimes_{\mathbb{Z}} \mathcal{O}_T$, $\mathcal{F}^p = \mathbb{R}^{2m-1} f_* \sigma_{\ge p} \Omega^\bullet_{X_T}$ and $\mathcal{J}^m = \mathcal{V}/\mathcal{F}^m + \mathbb{V}$. The Gauss–Manin connection ∇ induces a map $\overline{\nabla} : \mathcal{J}^m \to \mathcal{V}/\mathcal{F}^{m-1} \otimes \Omega^1_T$, whose kernel is denoted by $\mathcal{J}^m_h$; the global sections of this sheaf are called horizontal normal functions. If the conditions (i) and (ii) are satisfied, the first two cohomology sheaves of the complex of sheaves

$$F^m(\Omega^\bullet_T \otimes \mathbb{V}) : 0 \to \mathcal{F}^m \to \Omega^1_T \otimes \mathcal{F}^{m-1} \to \Omega^2_T \otimes \mathcal{F}^{m-2} \to \dots$$

are zero. The vanishing of $\mathcal{H}^1(T, F^m(\Omega^\bullet_T \otimes \mathbb{V}))$ shows that the infinitesimal invariant $\delta\nu$ associated to $\nu \in H^0(T, \mathcal{J}^m_h)$ vanishes, hence ν has locally constant liftings. If in addition $\mathcal{H}^0(T, F^m(\Omega^\bullet_T \otimes \mathbb{V}))$ vanishes, these locally constant sections are unique up to sections of the local system $\mathbb{V}$ and ν is a torsion section of $\mathcal{J}^m_h$; see [V1, Prop. 2.6]. Condition (iii) is vacuous if $Y = \mathbb{P}^{2m+r}$, because $E_1^{0,2m+1-b}(b) \subset H^{b,2m+1-b}(\mathbb{P}^{2m+r}) = 0$.

3 Proof of Nori's theorem

In this section we shall prove an effective version of Theorem 2.1 using Koszul cohomology computations. By Lemma 2.10 it suffices to show that certain E_∞ terms of the spectral sequence $E_r^{p,q}(b)$ introduced in Lemma 2.8 are zero for the base $T = U' = H^0(Y,E) \setminus \Delta'$. To this end, we shall identify the dual of the spectral sequence $E_r^{p,q}(b)$ with another spectral sequence ${}'E_r^{p,q}(b)$. The spectral sequence ${}'E_r^{p,q}(b)$ is one of the two spectral sequences associated to a double complex $B^{\bullet,\bullet}(b)$. Using the second spectral sequence ${}''E_r^{p,q}(b)$ associated to this double complex, whose E_1 terms are Koszul cohomology groups, we prove the vanishing of the relevant E_∞ terms.

We first consider the case of hypersurfaces. Let $(Y, \mathcal{O}_Y(1))$ be a smooth polarised variety of dimension $n+1$. Set $L = \mathcal{O}_Y(d_0)$, $V = H^0(Y,L)$. Let $X \in |L|$ be a smooth divisor. Set $S = \mathbb{P}H^0(Y,L)$ and let $\Delta \subset S$ and $\Delta' \subset V$ be the discriminant loci. Let $X_S \subset Y \times S$ be the universal family and let τ be the tautological section of the line bundle $\mathcal{L}_S = p_Y^*L \otimes p_S^*\mathcal{O}_S(1)$. Throughout the first part of this section we shall work with the base

$$T = U' = H^0(Y,L) \setminus \Delta'.$$

Recall from the proof of Lemma 2.10 that there exists a smooth morphism $h : T \to S$. Set $\mathcal{L} = (\mathrm{id} \times h)^*\mathcal{L}_S = p_Y^*L \otimes p_T^*\mathcal{O}_T(1)$. As T is a Zariski open subset of an affine space, the line bundle $\mathcal{O}_T(1)$ has a nowhere vanishing section. Hence $\mathcal{O}_T(1) \cong \mathcal{O}_T$ and $\mathcal{L} \cong p_Y^*L$. The tangent bundle to T is the trivial bundle $V \otimes \mathcal{O}_T$. Set $\sigma = (\mathrm{id} \times h)^*\tau \in H^0(Y_T, \mathcal{L})$. We have $X_S = V(\tau) \subset Y_S$ and $X_T = V(\sigma) \subset Y_T$. Choose a base point $s_0 \in T$ and define a map $i_0 : Y \to Y_T$ by $i_0(y) = (y, s_0)$.

Let $P^1(L)$ be the first jet bundle of L. A morphism $f : Y \to Z$ of smooth projective varieties induces a map $f^*P^1(L) \to P^1(f^*L)$ that fits into a commutative diagram (see [K] or [Pe, Prop. 3.4])

$$\begin{array}{ccccccccc}
0 & \to & f^*\Omega^1_Y \otimes f^*L & \to & f^*P^1(L) & \to & f^*L & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^1_Z \otimes f^*L & \to & P^1(f^*L) & \to & f^*L & \to & 0.
\end{array}$$

Let $\Sigma_{Y,L} = P^1(L)^\vee \otimes L$ be the bundle of first order differential operators on sections of L. It fits into an exact sequence of sheaves of $\mathcal{O}_Y$–modules

$$0 \to \mathcal{O}_Y \to \Sigma_{Y,L} \to T_Y \to 0$$

with extension class $e = 2\pi i.c_1(L)$ [A, pp. 195–196]. A morphism $f : Y \to Z$ induces a map $\Sigma_{X,f^*L} \to f^*\Sigma_{Y,L}$.

Lemma 3.1. *There exists a split exact sequence*

$$0 \to \Sigma_{Y,L} \to i_0^*\Sigma_{Y_T,\mathcal{L}} \to V \otimes \mathcal{O}_Y \to 0. \tag{3}$$

Proof: The morphism $i_0 : Y \to Y_T$ induces a homomorphism of vector bundles $f_1 : \Sigma_{Y,L} = \Sigma_{Y,i_0^*\mathcal{L}} \to i_0^*\Sigma_{Y_T,\mathcal{L}}$ that fits into a commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{O}_Y & \to & \Sigma_{Y,L} & \to & T_Y & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{O}_Y & \to & i_0^*\Sigma_{Y_T,\mathcal{L}} & \to & i_0^*T_{Y_T} & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & V \otimes \mathcal{O}_Y & = & V \otimes \mathcal{O}_Y & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0. & & \end{array}$$

The projection $p_Y : Y_T \to Y$ induces homomorphisms $\Sigma_{Y_T,\mathcal{L}} = \Sigma_{Y_T,p_Y^*L} \to p_Y^*\Sigma_{Y,L}$ and $f_2 : i_0^*\Sigma_{Y_T,\mathcal{L}} \to i_0^*p_Y^*\Sigma_{Y,L} = \Sigma_{Y,L}$. By functoriality, the composed map $f_2 \circ f_1 : \Sigma_{Y,L} \to \Sigma_{Y,L}$ is the identity, as it is induced by the map $\mathrm{id}_Y = p_Y \circ i_0 : Y \to Y$. Hence f_2 defines a splitting of the exact sequence. $\square$

Remark 3.2. There is a geometric interpretation of Lemma 3.1. The vector space $H^1(Y, \Sigma_{Y,L})$ parametrises infinitesimal deformations of the pair (Y, L) (cf. [SSU, Prop. (6.2)], [W, §1]). If $(\mathcal{Y}, \mathcal{L}, g, B)$ is a deformation of $(\mathcal{Y}_0, \mathcal{L}_0) \cong (Y, L)$ such that $g : \mathcal{Y} \to B$ is smooth, there exists a Kodaira–Spencer map

$$\rho_0 : T_0B \to H^1(Y_0, \Sigma_{Y,L})$$

that is induced by the middle column of the commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{O}_Y & \to & \Sigma_{Y,L} & \to & T_Y & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{O}_Y & \to & i_0^*\Sigma_{\mathcal{Y},\mathcal{L}} & \to & i_0^*T_{\mathcal{Y}} & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & T_0B \otimes \mathcal{O}_Y & = & T_0B \otimes \mathcal{O}_Y. & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

The Kodaira–Spencer map is given by cup product with the extension class of the middle column. If $\mathcal{Y} \cong Y \times S$ and $\mathcal{L} \cong p_Y^*L$ then the deformation is trivial. In this case the exact sequence in the middle column splits and the Kodaira–Spencer map is identically zero.

We recall some results about Koszul complexes. For every pair of natural numbers (p, q) there exists a contraction (or internal product) map

$$\begin{array}{rcl} \bigwedge^{p+q} V \otimes \bigwedge^q V^\vee & \to & \bigwedge^p V \\ x \otimes y \to & \to & \langle y, x \rangle \end{array}$$

that coincides with the duality pairing for $p = 0$. The map $\bigwedge^{p+q} V \to \bigwedge^p V$ given by contraction with $y \in \bigwedge^q V^\vee$ is the adjoint of the map $\bigwedge^p V^\vee \to \bigwedge^{p+q} V^\vee$ given by wedge product with y, i.e., we have the relation

$$\langle z, \langle y, x \rangle \rangle = \langle z \wedge y, x \rangle$$

for all $x \in \bigwedge^{p+q} V$, $z \in \bigwedge^p V^\vee$ (cf. [FH, Appendix B]). Let $R = \operatorname{Sym} V$ be the symmetric algebra on V. Choose bases $\{v_1, \ldots, v_N\}$ of V, $\{w_1, \ldots, w_N\}$ of $V^\vee$ that are dual to each other. The differentials of the Koszul complex

$$\bigwedge^N V \otimes R(-N) \to \ldots \to V \otimes R(-1) \to R$$

are the maps $\delta_{k+1} : \bigwedge^{k+1} V \otimes R(-k-1) \to \bigwedge^k V \otimes R(-k)$ defined by

$$\delta_{k+1}(v_{i_1} \wedge \ldots \wedge v_{i_{k+1}} \otimes y) = \sum_j (-1)^{j-1} v_{i_1} \wedge \ldots \widehat{v_{i_j}} \wedge \ldots \wedge v_{i_{k+1}} \otimes v_{i_j}.y.$$

These maps are given by contraction with the element $\omega = \sum_i v_i \otimes w_i \in V \otimes V^\vee$. If we sheafify the Koszul complex and restrict it to $Y \subset \mathbb{P}(V^\vee)$ we obtain a complex

$$\mathcal{K}^\bullet = (\bigwedge^N V \otimes L^{-N} \to \ldots \to V \otimes L^{-1} \to \mathcal{O}_Y)$$

that is concentrated in degrees $-N, \ldots, -1, 0$.

Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_Y$–modules.

Definition 3.3. **(Green)** [G2] *The Koszul cohomology group* $\mathcal{K}_{p,q}(\mathcal{F}, L) = H^{-p}(\Gamma(Y, \mathcal{K}^\bullet \otimes \mathcal{F} \otimes L^{p+q}))$ *is the cohomology group at the middle term of the complex*

$$\bigwedge^{p+1} V \otimes H^0(\mathcal{F} \otimes L^{q-1}) \to \bigwedge^p V \otimes H^0(\mathcal{F} \otimes L^q) \to \bigwedge^{p-1} V \otimes H^0(\mathcal{F} \otimes L^{q+1}).$$

Let M_L be the kernel of the surjective evaluation map $e_L : V \otimes \mathcal{O}_Y \to L$. The complex $\mathcal{K}^\bullet \otimes L^N$ can be obtained by taking the N^{th} exterior power of the short exact sequence of vector bundles

$$0 \to M_L \to V \otimes_{\mathbb{C}} \mathcal{O}_Y \to L \to 0. \tag{4}$$

The long exact sequence

$$\begin{aligned}0 \to \bigwedge^{p+1} M_L \otimes \mathcal{F} \otimes L^{q-1} \to \bigwedge^{p+1} V \otimes \mathcal{F} \otimes L^{q-1} \to \bigwedge^{p} V \otimes \mathcal{F} \otimes L^{q} \to \\ \to \bigwedge^{p-1} V \otimes \mathcal{F} \otimes L^{q+1} \to \ldots \to \mathcal{F} \otimes L^{p+q} \to 0\end{aligned}$$

obtained from (4) by taking exterior powers and twisting by $\mathcal{F} \otimes L^{q-1}$ shows that (cf. [G2])

$$\mathcal{K}_{p,q}(\mathcal{F}, L) = 0 \text{ if } H^1(Y, \bigwedge^{p+1} M_L \otimes \mathcal{F} \otimes L^{q-1}) = 0. \tag{5}$$

By Lemma 2.8 there exists a spectral sequence

$$E_1^{p,q}(b) = H^{p+q}(Y, \mathrm{Gr}_L^p\, i_0^* \Omega^b_{Y_T, X_T}) \Rightarrow H^{p+q}(Y, i_0^* \Omega^b_{Y_T, X_T}).$$

To prove Nori's theorem it suffices to verify the conditions of Lemma 2.10. By the Lefschetz hyperplane theorem $E_1^{p,q}(b) = 0$ if $b + q \le n$, so we may assume that $b + q \ge n + 1$. Set $N = \dim V$. Then $K_T \cong \bigwedge^N V^\vee \otimes \mathcal{O}_T$. As there is a nondegenerate pairing

$$\Omega^b_{Y_T, X_T} \otimes \Omega^{N+n+1-b}_{Y_T}(\log X_T) \to K_{Y_T}$$

we have an isomorphism

$$(\Omega^b_{Y_T, X_T})^\vee \cong K^{-1}_{Y_T} \otimes \Omega^{N+n+1-b}_{Y_T}(\log X_T).$$

By Serre duality we have

$$H^{p+q}(Y, i_0^* \Omega^b_{Y_T, X_T})^\vee \cong H^{n+1-p-q}(Y, \bigwedge^N V \otimes i_0^* \Omega^{N+n+1-b}_{Y_T}(\log X_T)).$$

Set

$$\mathcal{F}_b = \bigwedge^N V \otimes i_0^* \Omega^{N+n+1-b}_{Y_T}(\log X_T)$$

and define an increasing filtration $L^\bullet$ on $\bigwedge^N V$ as the trivial filtration starting in degree $-N$. The exact sequence

$$0 \to f^* \Omega^1_T \to \Omega^1_{Y_T}(\log X_T) \to \Omega^1_{Y_T/T}(\log X_T) \to 0$$

defines an increasing filtration $L^\bullet$ on $\Omega^\bullet_{Y_T}(\log X_T)$. The graded pieces of the induced filtration on the tensor product are

$$\begin{aligned}\mathrm{Gr}_L^{-p} \mathcal{F}_b &= \mathrm{Gr}_L^{-N} \bigwedge^N V \otimes \mathrm{Gr}_L^{N-p}\, i_0^* \Omega^{N+n+1-b}_{Y_T}(\log X_T) \\ &\cong \bigwedge^p V \otimes \Omega^{n+1-b+p}_Y(\log X_0),\end{aligned}$$

hence $E_1^{p,q}(b)^\vee \cong H^{n+1-p-q}(Y, \mathrm{Gr}_L^{-p} \mathcal{F}_b)$ and the spectral sequence dual to $E_r^{p,q}(b)$ is

$$\begin{aligned}E_1^{-p,n+1-q}(b) &= H^{n+1-p-q}(Y, \mathrm{Gr}_L^{-p} \mathcal{F}_b) \Rightarrow H^{n+1-p-q}(Y, \mathcal{F}_b) \\ E_1^{-x,y}(b) &\cong \bigwedge^x V \otimes H^{y-x}(Y, \Omega^{n+1-b+x}_Y(\log X_0)).\end{aligned} \tag{6}$$

Contraction with the 1–jet $j^1(\sigma)$ of $\sigma \in H^0(Y_T, \mathcal{L})$ gives rise to an exact sequence

$$0 \to T_{Y_T}(-\log X_T) \to \Sigma_{Y_T,\mathcal{L}} \to \mathcal{L} \to 0.$$

By taking exterior powers in the dual of this exact sequence we obtain for every $p = 0, \dots, N$ a complex

$$\mathcal{C}_p^\bullet(b) = (i_0^* \bigwedge^{n+2-b+p} \Sigma_{Y_T,\mathcal{L}}^\vee \otimes L \to \dots \to i_0^* K_{Y_T} \otimes \mathcal{L}^{N+b-p+1})$$

concentrated in degrees $0, \dots, N+b-p$, which is a resolution of the vector bundle $i_0^*\Omega_{Y_T}^{n+1-b+p}(\log X_T)$. The complex

$$\mathcal{C}^\bullet(b) = \bigwedge^N V \otimes \mathcal{C}_N^\bullet(b)$$

is a resolution of $\mathcal{F}_b$. Define a filtration $L^\bullet$ on $\mathcal{C}_N^\bullet(b)$ by

$$L^p\mathcal{C}_N^\bullet(b) = \operatorname{im}(\bigwedge^p V^\vee \otimes \mathcal{C}_{N-p}^\bullet(b) \to \mathcal{C}_N^\bullet(b)).$$

The induced filtration on $\mathcal{C}^\bullet(b)$ is $L^{-p}\mathcal{C}^\bullet(b) = \bigwedge^N V \otimes L^{N-p}\mathcal{C}_N^\bullet(b)$. As the morphism $\mathcal{F}_b \to \mathcal{C}^\bullet(b)$ is compatible with the filtrations, we obtain a resolution of $\operatorname{Gr}_L^{-p} \mathcal{F}_b$ by the complex

$$Gr_L^{-p}\mathcal{C}^\bullet(b) = (\bigwedge^p V \otimes \bigwedge^{n+2-b+p} \Sigma_{Y,L}^\vee \otimes L \to \dots \to \bigwedge^p V \otimes K_Y \otimes L^{b-p+1})$$

concentrated in degrees $0, \dots, b-p$.

Let $\mathcal{B}^{\bullet,\bullet}(b)$ be the subcomplex of the double complex $\mathcal{K}^\bullet \otimes \operatorname{Gr}_L^0 \mathcal{C}^\bullet(b)$ that consists of terms of nonnegative total degree. (This double complex appeared already in [GM2].) We have

$$\mathcal{B}^{-i,j}(b) = \bigwedge^i V \otimes \bigwedge^{n+2-b+j} \Sigma_{Y,L}^\vee \otimes L^{j-i+1}, \quad j - i \geq 0.$$

The complex $\mathcal{B}^{\bullet,\bullet}(b)$ is a second quadrant double complex which consists of the terms $\mathcal{B}^{-i,j}(b)$ with $0 \leq i \leq b$, $0 \leq j \leq b$ and $j - i \geq 0$:

$$\begin{array}{ccccc}
\bigwedge^b V \otimes K_Y \otimes L & \to & \dots & \to & K_Y \otimes L^{b+1} \\
 & & & & \uparrow \\
 & \ddots & & & \vdots \\
 & & \ddots & & \uparrow \\
 & & & & \bigwedge^{n+2-b} \Sigma_{Y,L}^\vee \otimes L.
\end{array}$$

Consider the total complex $\mathcal{B}^\bullet(b) = s(\mathcal{B}^{\bullet,\bullet}(b))$ associated to the double complex $\mathcal{B}^{\bullet,\bullet}(b)$. We have

$$\begin{aligned}
\mathcal{B}^k(b) &= \bigoplus_{i=0}^b \mathcal{B}^{-i,k+i}(b) \\
&= \bigoplus_{i=0}^b \bigwedge^i V \otimes \bigwedge^{n+2-b+k+i} \Sigma_{Y,L}^\vee \otimes L^{k+1}.
\end{aligned}$$

Define a filtration $'F^\bullet$ on $\mathcal{B}^\bullet(b)$ by

$$'F^{-p}\mathcal{B}^k(b) = \bigoplus_{i=0}^p \mathcal{B}^{-i,k+i}(b).$$

We define the horizontal differential $d_{\mathrm{hor}} : \mathcal{B}^{-p,q}(b) \to \mathcal{B}^{-p+1,q}(b)$ using the differential of the Koszul complex: $d_{\mathrm{hor}} = \delta_p \otimes \mathrm{id}$. The vertical differential $d_{\mathrm{vert}} : \mathcal{B}^{-p,q}(b) \to \mathcal{B}^{-p,q+1}(b)$ is given by contraction with the 1–jet $j^1(s_0) \in P^1(L)$. As the horizontal and vertical differentials commute, we define a modified vertical differential d'_{vert} by $d'_{\mathrm{vert}}|_{\mathcal{B}^{-p,q}(b)} = (-1)^{N-p} d_{\mathrm{vert}}$. The differentials d_{hor} and d'_{vert} of the complex $\mathcal{B}^\bullet(b)$ anticommute. We define a differential d of $\mathcal{B}^\bullet(b)$ by $d = d_{\mathrm{hor}} + d'_{\mathrm{vert}}$.

The splitting of the exact sequence (3) induces an isomorphism of vector bundles

$$i_0^* \bigwedge^{n+2-b+p+N} \Sigma^\vee_{Y_T,\mathcal{L}} \to \bigoplus_{k=0}^{b-p} \bigwedge^{N-k} V^\vee \otimes \bigwedge^{n+2-b+p+k} \Sigma^\vee_{Y,L}.$$

Using the contraction maps $\bigwedge^N V \otimes \bigwedge^{N-k} V^\vee \to \bigwedge^k V$, we obtain an isomorphism of vector bundles

$$\bigwedge^N V \otimes i_0^* \bigwedge^{n+2-b+p+N} \Sigma^\vee_{Y_T,\mathcal{L}} \to \bigoplus_{k=0}^{b-p} \bigwedge^k V \otimes \bigwedge^{n+2-b+p+k} \Sigma^\vee_{Y,L}$$

and an isomorphism of vector bundles $h_p : \mathcal{C}^p(b) \to \mathcal{B}^p(b)$, which is a direct sum of homomorphisms $h_{p,k} : \mathcal{C}^p(b) \to \mathcal{B}^{-k,k+p}(b)$.

Lemma 3.4. *The maps h_p induce an isomorphism $h : \mathcal{C}^\bullet(b) \to \mathcal{B}^\bullet(b)$ of complexes of sheaves of $\mathcal{O}_Y$–modules that is compatible with the filtrations $L^\bullet$ and $'F^\bullet$.*

Proof: Choose an open covering $\mathcal{U} = \{U_i\}_{i\in I}$ of Y and trivialisations of L and the bundle of differential operators over U_i. The variety Y_T is covered by open subsets $U \times T$ ($U \in \mathcal{U}$) with local coordinates $(y_1, \ldots, y_{n+1}, t_1, \ldots, t_N)$. Let $\{dt_1, \ldots, dt_N\}$ be a basis of $V^\vee \cong \Omega^1_{T,0}$, and let $\{s_1, \ldots, s_N\}$ be the dual basis of V. We write $f_i = s_i|_U \in \Gamma(U, L) \cong \Gamma(U, \mathcal{O}_U)$. We have

$$\begin{aligned} \sigma|_{U\times T} &= \sum_k t_k f_k(y) \\ j^1(\sigma|_{U\times T}) &= \sum_k t_k f_k + \sum_{k,i} t_k \frac{\partial f_k(y)}{\partial y_i} \otimes dy_i + \sum_i f_i \otimes dt_i. \end{aligned}$$

As $i_0^*\sigma|_{U\times T} = s_0|_U = f_0$, we have $i_0^* j^1(\sigma|_{U\times T}) = \omega_1 + \omega_2$ with

$$\begin{aligned} \omega_1 &= f_0 + \sum_i \frac{\partial f_0}{\partial y_i} \otimes dy_i = j^1(f_0) \\ \omega_2 &= \sum_i f_i \otimes dt_i. \end{aligned}$$

The differential d' of the complex $\mathcal{C}^\bullet(b)$ is given by contraction with $i_0^* j^1(\sigma)$. Let d'_i be the map given by contraction with ω_i for $i = 1, 2$. The differential d' splits as $d' = d'_1 + d'_2$. Put $m = n+2-b+p+N$. To show that $h_{p+1} \circ d' = d \circ h_p$, we shall show that the diagrams

$$\begin{array}{ccc} \bigwedge^k V \otimes \bigwedge^{m-N+k} \Sigma^\vee_{Y,L} \otimes L^{p+1} & \stackrel{d'_{\mathrm{vert}}}{\to} & \bigwedge^k V \otimes \bigwedge^{m+1-N+k} \Sigma^\vee_{Y,L} \otimes L^{p+2} \\ \uparrow h_{p,k} & & \uparrow h_{p+1,k} \\ \bigwedge^N V \otimes i_0^* \bigwedge^m \Sigma^\vee_{Y_T,\mathcal{L}} \otimes L^{p+1} & \stackrel{d'_1}{\to} & \bigwedge^N V \otimes i_0^* \bigwedge^{m+1} \Sigma^\vee_{Y_T,\mathcal{L}} \otimes L^{p+2} \end{array}$$

and

$$\begin{array}{ccc} \bigwedge^k V \otimes \bigwedge^{m-N+k} \Sigma^\vee_{Y,L} \otimes L^{p+1} & \stackrel{d_{\mathrm{hor}}}{\to} & \bigwedge^{k-1} V \otimes \bigwedge^{m-N+k} \Sigma^\vee_{Y,L} \otimes L^{p+2} \\ \uparrow h_{p,k} & & \uparrow h_{p+1,k-1} \\ \bigwedge^N V \otimes i_0^* \bigwedge^m \Sigma^\vee_{Y_T,\mathcal{L}} \otimes L^{p+1} & \stackrel{d'_2}{\to} & \bigwedge^N V \otimes i_0^* \bigwedge^{m+1} \Sigma^\vee_{Y_T,\mathcal{L}} \otimes L^{p+2} \end{array}$$

commute for every $p \geq 1$ and $k \in \{0, \ldots, b-p\}$.

Given $\zeta \in i_0^* \bigwedge^{n+2-b+p+N} \Sigma^\vee_{Y_T,\mathcal{L}} \otimes L^{p+1}|_U$, write

$$\zeta = \eta + f \otimes \eta'$$

with $\eta \in \Omega^{n+2-b+p+N}_{Y_T} \otimes L^{p+1}|_U$, $f \in \mathcal{O}_U$ and $\eta' \in i_0^* \Omega^{n+1-b+p+N}_{Y_T} \otimes L^{p+1}|_U$. We shall check that the two diagrams commute for η; the proof for $f \otimes \eta'$ is similar. Write $\eta = \sum_k \eta_{1,k} \wedge \eta_{2,k} \otimes g_k$ with $\eta_{1,k} \in \bigwedge^{N-k} V^\vee \otimes \mathcal{O}_U$, $\eta_{2,k} \in \bigwedge^{n+2-b+p+k} \Sigma^\vee_{Y,L}|_U$, $g_k \in L^{p+1}|_U \cong \mathcal{O}_U$.

In the first diagram we have

$$\begin{aligned} d'_1(\xi \otimes \eta) &= \xi \otimes \omega_1.\eta \\ &= \xi \otimes f_0 \eta + \\ &\quad \xi \otimes \sum_{k,i} dy_i \wedge \eta_{1,k} \wedge \eta_{2,k} \otimes g_k \otimes \frac{\partial f_0}{\partial y_i} \\ &= \xi \otimes f_0 \eta + \\ &\quad (-1)^{N-k} \xi \otimes \sum_{k,i} \eta_{1,k} \wedge dy_i \wedge \eta_{2,k} \otimes g_k \otimes \frac{\partial f_0}{\partial y_i} \end{aligned}$$

and hence

$$\begin{aligned} h_{p+1,k}(d'_1(\xi \otimes \eta)) &= \sum_k \langle \eta_{1,k}, \xi \rangle \wedge \eta_{2,k} \otimes g_k \otimes f_0 + \\ &\quad (-1)^{N-k} \sum_{k,i} \langle \eta_{1,k}, \xi \rangle \otimes dy_i \wedge \eta_{2,k} \otimes g_k \otimes \frac{\partial f_0}{\partial y_i}. \end{aligned}$$

As
$$d'_{\mathrm{vert}}(h_{p,k}(\xi \otimes \eta)) = d'_{\mathrm{vert}}(\sum_k \langle \eta_{1,k}, \xi\rangle \otimes \eta_{2,k} \otimes g_k)$$
and $j^1(f_0) = \omega_1$, the first diagram commutes.

In the second diagram we have
$$h_{p+1,k-1}(d'_2(\xi \otimes \eta)) = \sum_{k,i} \langle dt_i \wedge \eta_{1,k}, \xi\rangle \otimes \eta_{2,k} \otimes f_i \otimes g_k$$
and
$$\begin{aligned} d_{\mathrm{hor}}(h_{p,k}(\xi \otimes \eta)) &= \sum_{k,i} \langle dt_i, \langle \eta_{1,k}, \xi\rangle\rangle \otimes \eta_{2,k} \otimes f_i \otimes g_k \\ &= \sum_{k,i} \langle dt_i \wedge \eta_{1,k}, \xi\rangle \otimes \eta_{2,k} \otimes f_i \otimes g_k. \end{aligned}$$
Hence the second diagram commutes. The isomorphism of complexes h is compatible with the filtrations because the sequence (3) splits.

$\square$

Put $B^{\bullet,\bullet}(b) = \Gamma(Y, \mathcal{B}^{\bullet,\bullet}(b))$ and $B^\bullet(b) = s(B^{\bullet,\bullet}(b))$. We have two spectral sequences
$$'E_1^{p,q}(b) = H^q(B^{p,\bullet}(b)) \Rightarrow H^{p+q}(B^\bullet(b)) \tag{7}$$
$$''E_1^{p,q}(b) = H^q(B^{\bullet,p}(b)) \Rightarrow H^{p+q}(B^\bullet(b)). \tag{8}$$

The E_1 terms of the first spectral sequence are related to Jacobi rings and the E_1 terms of the second spectral sequence are Koszul cohomology groups. We start with the description of $'E_1^{-p,q}(b)$. Let $\mathcal{F}$ be a coherent sheaf on Y and let
$$R_{Y,s_0}(\mathcal{F}) = \mathrm{coker}\,(H^0(Y, \mathcal{F} \otimes \Sigma_L \otimes L^{-1}) \to H^0(Y, \mathcal{F}))$$
be the cokernel of the map given by contraction with the 1–jet $j^1(s_0) \in P^1(L)$ (this is a graded piece of the Jacobi ring defined by Green [G1]). We have
$$'E_1^{-p,b}(b) \cong \bigwedge^p V \otimes R_{Y,s_0}(K_Y \otimes L^{b-p+1}). \tag{9}$$

As $B^{-q,p}(b) = \bigwedge^q V \otimes H^0(Y, \bigwedge^{n+2-b+p} \Sigma^\vee_{Y,L} \otimes L^{p-q+1})$, it follows that
$$''E_1^{p,-q}(b) = \mathcal{K}_{q,p-q+1}(\bigwedge^{n+2-b+p} \Sigma^\vee_{Y,L}, L). \tag{10}$$

Lemma 3.5. *For all $t \in H^0(Y, L) \setminus \Delta'$ there exists a morphism of spectral sequences*

$$\psi : {}'E_r^{-p,n+1-q}(b) \to E_r^{-p,n+1-q}(b).$$

Proof: Let $g : (\mathcal{F}_b, L) \to (\mathcal{B}^\bullet(b), {}'F)$ be the composition of the filtered quasi–isomorphism $(\mathcal{F}_b, L) \to (\mathcal{C}^\bullet(b), L)$ and the isomorphism $h : (\mathcal{C}^\bullet(b), L) \to (\mathcal{B}^\bullet(b), {}'F)$. There exists a spectral sequence

$$E_1^{-p,n+1-q}(\mathcal{B}^\bullet(b), {}'F) = \mathbb{H}^{n+1-p-q}(\mathrm{Gr}_{'F}^{-p} \mathcal{B}^\bullet(b)) \Rightarrow \mathbb{H}^{n+1-p-q}(\mathcal{B}^\bullet(b)).$$

The morphism g induces an isomorphism of spectral sequences

$$E_r^{-p,n+1-q}(\mathcal{F}_b, L) \cong E_r^{-p,n+1-q}(\mathcal{B}^\bullet(b), {}'F).$$

Let $(\mathcal{I}^\bullet, G)$ be a filtered injective resolution of $(\mathcal{B}^\bullet(b), {}'F)$. The morphism of complexes $\Gamma(Y, \mathcal{B}^\bullet(b)) \to \Gamma(Y, \mathcal{I}^\bullet)$ induces a morphism of spectral sequences ${}'E_r{}^{-p,n+1-q}(b) \to E_r^{-p,n+1-q}(\mathcal{B}^\bullet(b), {}'F)$. The composition of this morphism with the inverse of the isomorphism $E_r^{-p,n+1-q}(b) \to E_r^{-p,n+1-q}(\mathcal{B}^\bullet(b), {}'F)$ is a morphism of spectral sequences

$$\psi : {}'E_r^{-p,n+1-q}(b) \to E_r^{-p,n+1-q}(b).$$

□

Lemma 3.6. *Suppose that for all $k \le c$ and for all $t \in U' = H^0(Y, L) \setminus \Delta'$ we have*

(i) $E_1^{-x,y}(b) \cong {}'E_1^{-x,y}(b)$ for all (x, y) such that $0 \le x \le b$, $b-k+1 \le y \le b$;

(ii) ${}''E_1^{x,-y}(b) = 0$ for all (x, y) such that $x \ge 0$, $y \ge 0$ and $b - k + 1 \le x - y \le b$.

Then for every smooth morphism $g : T \to U$ we have $H^{n+k}(Y_T, X_T) = 0$ for all $k \le c$.

Proof: By Lemma 2.10 it suffices to show that for all $t \in U'$ we have $E_\infty^{p,q}(b) = 0$ for all $k \le c$ and for all (p, q, b) such that $p + q + b \le n + k$, $b \ge b_k$. By Serre duality we have

$$E_\infty^{p,q}(b) = H^{p+q}(Y, i_0^* \Omega^b_{Y_T, X_T}) = 0 \Longleftrightarrow E_\infty^{-p,n+1-q}(b) = H^{n+1-p-q}(Y, \mathcal{F}_b) = 0.$$

As $p + q + b \le n + k$ and $b + q \ge n + 1$ we have $b - k + 1 \le n + 1 - q \le b$ and $0 \le p \le k - 1$. By Lemma 3.5 we have

$$E_\infty^{-p,n+1-q}(b) \cong {}'E_\infty^{-p,n+1-q}(b)$$

if condition (i) is satisfied. (This follows by consideration of the incoming and outgoing differentials at the positions $(-p, n+1-q)$; note that $E_1^{-x,y}(b) = 0$ if $x > b$.) As ${}'E_\infty^{n+1-p-q}(b) \cong {}''E_\infty^{n+1-p-q}(b)$, it follows that ${}'E_\infty^{-p,n+1-q}(b) = 0$ if ${}''E_\infty^{x,-y}(b) = 0$ for all (x, y) such that $b - k + 1 \le x - y \le b$. This follows from condition (ii). □

We turn to the case of complete intersections. Let $(Y, \mathcal{O}_Y(1))$ be a smooth polarised variety of dimension $n+r+1$, and set $E = \mathcal{O}_Y(d_0) \oplus \cdots \oplus \mathcal{O}_Y(d_r)$. Let $P = \mathbb{P}(E^\vee)$ be the projective bundle whose fiber over $y \in Y$ is the projective space of hyperplanes in E_y, and let $\pi : P \to Y$ be the projection map. The tautological line bundle over P will be denoted by ξ_E. The zero locus $X_t = V(t)$ of a general section $t \in H^0(Y, E)$ is a smooth complete intersection of dimension n and multidegree $(d_0, \ldots, d_r)$ in Y. Let $\tilde{t} \in H^0(P, \xi_E)$ be the section that corresponds to t under the canonical isomorphism $H^0(Y, E) = H^0(P, \xi_E)$. Set $\tilde{X}_t = V(\tilde{t}) \subset P$.

There is a relative version of this construction. Set $S = \mathbb{P}H^0(Y, E)$ and define

$$\mathcal{E} = p_Y^* E \otimes p_S^* \mathcal{O}_S(1).$$

For a smooth morhpism $h : T \to S$ we write $\mathcal{E}_T = (\mathrm{id} \times h)^* \mathcal{E}$ and $P_T = \mathbb{P}(\mathcal{E}_T^\vee)$. Note that in general P_T is not isomorphic to $P \times T$, unless the line bundle $\mathcal{O}_T(1)$ is trivial. Let $\pi_T : P_T \to Y_T$ be the projection map and let ξ_T be the tautological line bundle on P_T. The universal family $X_S \subset Y_S$ is the zero locus of the tautological section $\tau \in H^0(Y_S, \mathcal{E})$. Set $\sigma = (\mathrm{id} \times h)^* \tau \in H^0(Y_T, \mathcal{E}_T)$. We have $X_T = V(\sigma) \subset Y_T$. Let $\tilde{\sigma} \in H^0(P_T, \xi_T)$ be the section that corresponds to σ and define $\tilde{X}_T = V(\tilde{\sigma}) \subset P_T$.

Lemma 3.7. *The map $(\pi_T)_*$ induces an isomorphism*

$$H^{k+2r}(P_T, \tilde{X}_T) \cong H^k(Y_T, X_T)$$

for all $k \geq 0$.

Proof: (cf. [ENS, 2.1]) Consider the diagram

$$\begin{array}{ccccccc}
\pi_T^{-1}(X_T) = \mathbb{P}(\mathcal{E}_T^\vee|_{X_T}) & \subset & \tilde{X}_T & \subset & P_T & \supset & P_T \setminus \tilde{X}_T \\
 & & & & \downarrow{\scriptstyle \pi_T} & & \downarrow{\scriptstyle \pi_T} \\
 & & X_T & \subset & Y_T & \supset & Y_T \setminus X_T.
\end{array}$$

As the line bundle ξ_T restricts to $\mathcal{O}_{\mathbb{P}}(1)$ on each fiber of π_T, the induced map

$$\pi_T : P_T \setminus \tilde{X}_T \to Y_T \setminus X_T$$

is a fiber bundle with fibers isomorphic to $\mathbb{A}^r$. Hence $(\pi_T)_*$ induces an isomorphism

$$H_c^{k+2r}(P_T \setminus \tilde{X}_T) \cong H_c^k(Y_T \setminus X_T).$$

By Poincaré–Lefschetz duality we find an isomorphism $H^{k+2r}(P_T, \tilde{X}_T) \cong H^k(Y_T, X_T)$.

$\square$

Remark 3.8. In a similar way one shows that $H^k(Y, X_t) \cong H^{k+2r}(P, \tilde{X}_t)$ for all $t \in T$. As this map is an isomorphism of mixed Hodge structures, we obtain isomorphisms $H^a(Y, \Omega^b_{Y,X_t}) \cong H^{a+r}(P, \Omega^{b+r}_{P,\tilde{X}_t})$ (cf. [DD, Lemme 2.2]).

Let $\iota_t : P_t \to P_T$ be the inclusion map. As in Lemma 2.8 one constructs a spectral sequence

$$\tilde{E}_1^{p,q}(b) = \Omega^p_{T,t} \otimes H^{p+q}(P_t, \Omega^{b-p}_{P_t,\tilde{X}_t}) \Rightarrow H^{p+q}(P_t, \iota_t^* \Omega^b_{P_T,\tilde{X}_T}).$$

We have

$$\begin{aligned} E_1^{p,q}(b) &= \Omega^p_{T,t} \otimes H^{p+q}(Y, \Omega^{b-p}_{Y,X_t}) \\ &\cong \Omega^p_{T,t} \otimes H^{p+q+r}(P, \Omega^{b-p+r}_{P,\tilde{X}_t}) = \tilde{E}_1^{p,q+r}(b+r). \end{aligned}$$

Lemma 3.9. *Suppose that for all $k \le c$ and for all $t \in U' = H^0(Y, E) \setminus \Delta'$ we have*

(i) $\tilde{E}_1^{-x,y+r}(b+r) \cong {}'\tilde{E}_1^{-x,y+r}(b+r)$ for all (x, y) such that $0 \le x \le b$ and $b - k + 1 \le y \le b$;

(ii) ${}''\tilde{E}_1^{x,-y+r}(b+r) = 0$ for all (x, y) such that $x \ge 0$, $y \ge r$ and $b-k+1 \le x - y \le b$.

Then for every smooth morphism $g : T \to U$ we have $H^{n+k}(Y_T, X_T) = 0$ for all $k \le c$.

Proof: By Lemma 3.7 we have

$$H^{n+k}(Y_T, X_T) = 0 \Longleftrightarrow H^{n+k+2r}(P_T, \tilde{X}_T) = 0.$$

As U' is an open subset of the affine space $H^0(Y, E)$, the line bundle $\mathcal{O}_{U'}(1)$ is trivial. Hence $P_{U'} \cong P \times U'$. It follows from Lemma 3.6, applied to the pair $(Y, L) = (P, \xi_E)$, that the conditions of Lemma 2.10 are satisfied for the pair (P, ξ_E). (In the proof of Lemma 3.6 we have to replace $n = \dim X_t$ by $n' = n + 2r = \dim \tilde{X}_t$ and b by $b' = b + r$.) The result follows from Lemma 2.10, applied to the pair $(P_U, \tilde{X}_U)$. Note that the conclusion of Lemma 2.6 (which is used in the proof of Lemma 2.10) remains valid for the pair $(P_U, \tilde{X}_U)$, although $P_U \not\cong P \times U$, by Remark 2.11. $\square$

Let M_ξ be the kernel of the surjective evaluation map $e : V \otimes \mathcal{O}_P \to \xi_E$.

Lemma 3.10.

(i) $H^i(P, \Omega_P^j \otimes \xi_E^k) = 0$ *if*

$$H^{i+t}(Y, \Omega_Y^u \otimes \bigwedge^v E \otimes S^w E) = 0$$

for all (t, u, v, w) *such that* $0 \le u \le j$, $u + v = j + t + 1$, $v + w = k$ *and* $0 \le t \le r - j + u$.

(ii) $H^1(P, \bigwedge^i M_\xi \otimes \Omega_P^j \otimes \xi_E^k) = 0$ *if*

$$H^{s+t+1}(Y, \bigwedge^e M_E \otimes \bigwedge^f E \otimes \Omega_Y^u \otimes \bigwedge^v E \otimes S^w E) = 0$$

for all (e, f, s, t, u, v, w) *such that* $s \ge 0$, $t \ge 0$, $e + f = i + s + 1$, $0 \le e \le i$, $u + v = j + t + 1$, $v + w = k - s - 1$.

Proof: (i): Let $\pi : P \to Y$ be the projection map. The exact sequence

$$0 \to \pi^*\Omega_Y^1 \to \Omega_P^1 \to \Omega_{P/Y}^1 \to 0 \tag{11}$$

defines a filtration $L^\bullet$ on Ω_P^j with graded pieces $\mathrm{Gr}_L^u \Omega_P^j = \pi^*\Omega_Y^u \otimes \Omega_{P/Y}^{j-u}$. Hence it suffices to show that $H^i(P, \pi^*\Omega_Y^u \otimes \Omega_{P/Y}^{j-u} \otimes \xi_E^k) = 0$ for all $0 \le u \le j$. From the exact sequence

$$0 \to \mathcal{O}_P \to \pi^*E^\vee \otimes \xi_E \to T_{P/Y} \to 0$$

we obtain a resolution

$$\pi^*\bigwedge^{r+1} E \otimes \xi_E^{-r-1} \to \ldots \to \pi^*\bigwedge^{j-u+1} E \otimes \xi_E^{-j+u-1} \to \Omega_{P/Y}^{j-u} \to 0. \tag{12}$$

Using this resolution, we find that it suffices to show that

$$H^{i+t}(P, \pi^*\Omega_Y^u \otimes \pi^*\bigwedge^{j-u+t+1} E \otimes \xi_E^{k-j+u-t-1}) = 0$$

for all $0 \le t \le r - j + u$. Using the projection formula and the Leray spectral sequence, we find the condition of the lemma if we set $v = j - u + t + 1$ and $w = k - j + u - t - 1$.

(ii): From the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & \Omega_{P/Y}^1 \otimes \xi_E & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \to & \pi^*M_E & \to & V \otimes \mathcal{O}_P & \to & \pi^*E & \to & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \to & M_\xi & \to & V \otimes \mathcal{O}_P & \to & \xi_E & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & Q & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0. & & & & & &
\end{array}$$

and the snake lemma we deduce an exact sequence

$$0 \to \pi^* M_E \to M_\xi \to \Omega^1_{P/Y} \otimes \xi_E \to 0.$$

This exact sequence induces a filtration on $\bigwedge^i M_E$. Using this filtration, the filtration on Ω^j_Y coming from (11) and the resolution (12) we find that $H^1(P, \bigwedge^i M_\xi \otimes \Omega^j_P \otimes \xi^k_E) = 0$ if

$$H^{s+t+1}(Y, \bigwedge^e M_E \otimes \bigwedge^f E \otimes \Omega^u_Y \otimes \bigwedge^v E \otimes S^w E) = 0$$

where $f = i - e + s + 1$, $v = j - u + t + 1$ and $w = k - j + u - s - t - 2$. $\square$

Lemma 3.11. **(Green)** *Let E be a vector bundle on $\mathbb{P}^N$ and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_{\mathbb{P}}$–modules. Then $m(E \otimes \mathcal{F}) \leq m(E) + m(\mathcal{F})$ and $m(\bigwedge^c E) \leq m(E^{\otimes c})$.*

Proof: This follows from the proof of [G3, Lemma 1]. $\square$

Recall the notation

$$m_j = m(\Omega^j_Y) = \min\{k \in \mathbb{Z} | H^i(Y, \Omega^j_Y(k - i)) = 0 \text{ for all } i > 0\}.$$

Lemma 3.12.

$$H^i(Y, \Omega^j_Y \otimes \bigwedge^c M_E \otimes \mathcal{O}_Y(k)) = 0$$

if $i \geq 1$ and $k + i \geq m_j + c$.

Proof: Define $V = H^0(Y, \mathcal{O}_Y(1))$. The line bundle $\mathcal{O}_Y(1)$ defines an embedding

$$f : Y \to \mathbb{P}(V^\vee).$$

Set

$$W = \oplus_{i=0}^r S^{d_i} V$$

and let M_W be the kernel of the surjective map $W \otimes \mathcal{O}_Y \to E$. Paranjape [Pa, Lemma 2.4] proves that

$$H^i(Y, \Omega^j_Y \otimes \bigwedge^c M_W \otimes \mathcal{O}_Y(k)) = 0$$

if $i \geq 1$ and $k + i \geq m_j + c$. The idea is to write the vector bundle M_W on Y as a direct sum of pullbacks of vector bundles E_i on $\mathbb{P}(V^\vee)$ that are 1–regular and to apply the projection formula to reduce to a vanishing statement on projective space. The latter statement is proved by applying Lemma 3.11 to the 1–regular vector bundle $\oplus_{i=0}^r E_i$ and the coherent sheaf $f_* \Omega^j_Y$.

The vector bundle M_E differs from Paranjape's vector bundle M_W. Define $K = \oplus_{i=0}^{r} H^0(\mathbb{P}(V^\vee), \mathcal{I}_Y(d_i))$ and $Q = \oplus_{i=0}^{r} H^1(\mathbb{P}(V^\vee), \mathcal{I}_Y(d_i))$. There is an exact sequence

$$0 \to K \to W \to H^0(Y, E) \to Q \to 0.$$

We have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K \otimes \mathcal{O}_Y & = & K \otimes \mathcal{O}_Y & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & M_W & \to & W \otimes \mathcal{O}_Y & \to & E & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & M_E & \to & H^0(Y, E) \otimes \mathcal{O}_Y & \to & E & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & Q \otimes \mathcal{O}_Y & = & Q \otimes \mathcal{O}_Y & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

The exact sequence of vector bundles

$$0 \to K \otimes \mathcal{O}_Y \to M_W \to M_E \to Q \otimes \mathcal{O}_Y \to 0$$

can be split into two exact sequences of vector bundles

$$0 \to K \otimes \mathcal{O}_Y \to M_W \to \mathcal{R} \to 0 \tag{13}$$
$$0 \to \mathcal{R} \to M_E \to Q \otimes \mathcal{O}_Y \to 0. \tag{14}$$

The exact sequence (14) induces a filtration on $\bigwedge^c M_E$ with graded pieces

$$\bigwedge^p \mathcal{R} \otimes \bigwedge^{c-p} Q \otimes \mathcal{O}_Y.$$

Hence

$$H^i(Y, \Omega_Y^j \otimes \bigwedge^c M_E \otimes \mathcal{O}_Y(k)) = 0$$

if

$$\bigwedge^{c-p} Q \otimes H^i(Y, \Omega_Y^j \otimes \bigwedge^p \mathcal{R} \otimes \mathcal{O}_Y(k)) = 0$$

for $p = 0, \cdots, c$. The exact sequence (13) induces a long exact sequence (cf. [G5, Lecture 4, p. 39])

$$0 \to S^p K \otimes \mathcal{O}_Y \to S^{p-1} K \otimes M_W \to S^{p-2} K \otimes \bigwedge^2 M_W \to \cdots \to$$
$$\to \bigwedge^p M_W \to \bigwedge^p \mathcal{R} \to 0$$

for all $p \geq 0$. Using this exact sequence we find that

$$H^i(Y, \Omega_Y^j \otimes \bigwedge^p \mathcal{R} \otimes \mathcal{O}_Y(k)) = 0$$

if

$$S^tK \otimes H^{i+t}(Y, \Omega^j_Y \otimes \bigwedge^{p-t} M_W \otimes \mathcal{O}_Y(k)) = 0$$

for all $t = 0, \cdots, p$. Using [loc.cit., Lemma 2.4] we find the condition

$$k + i + t \geq m_j + p - t.$$

Hence

$$H^i(Y, \Omega^j_Y \otimes \bigwedge^c M_E \otimes \mathcal{O}_Y(k)) = 0$$

if $i \geq 1$ and $k + i \geq m_j + c$.

$\square$

For an increasing multi–index $I = (i_1, \ldots, i_p)$, $0 \leq i_1 \leq \ldots \leq i_p \leq r$, we set $d^{(I)} = \sum_{k=1}^p d_{i_k}$. For a strictly increasing multi–index $I = (i_1, \ldots, i_p)$, $0 \leq i_1 < \ldots < i_p \leq r$, we write $d^{<I>}$ in stead of $d^{(I)}$.

Theorem 3.13. *Let $(Y, \mathcal{O}_Y(1))$ be a smooth polarised variety of dimension $n + r + 1$. Let $d_0, \ldots, d_r$ be natural numbers ordered in such a way that $d_0 \geq \cdots \geq d_r$. Define $E = \mathcal{O}_Y(d_0) \oplus \ldots \mathcal{O}_Y(d_r)$ and let $U \subset \mathbb{P}H^0(Y, E)$ be the complement of the discriminant locus. Let m_j be the regularity of Ω^j_Y and define*

$$m_Y = \max\{m_j - j - 1 | 0 \leq j \leq \dim\, Y\}.$$

Set $\mu = [\frac{n+c}{2}]$. Consider the conditions

(C) $\sum_{\nu=\min(c,r)}^r d_\nu \geq m_Y + \dim\, Y - 1$;

(C_i) $\sum_{\nu=i}^r d_\nu + (\mu - c + i)d_r \geq m_Y + \dim\, Y + c - i$.

If condition (C) is satisfied and if the conditions (C_i) are satisfied for all i with $0 \leq i \leq \min(c-1, r)$, then for every smooth morphism $g : T \to U$ we have $H^{n+k}(Y_T, X_T) = 0$ for all $k \leq c$.

Proof: We shall verify the conditions (i) and (ii) of Lemma 3.9.

(i): Set $n' = n + 2r$ and $b' = b + r$. From (6) and (7), applied with (Y, L) replaced by (P, ξ) and (n, b) replaced by (n', b'), we obtain

$$\begin{aligned} \tilde{E}_1^{-x,y+r}(b+r) &= \bigwedge^x V \otimes H^{y+r-x}(P, \Omega_P^{n'+1-b'+x}(\log \tilde{X}_t)) \\ {}'\tilde{E}_1^{-x,y+r}(b+r) &= H^{y+r}(\tilde{B}^{-x,\bullet}(b+r)) \end{aligned}$$

where $\tilde{B}^{-x,\bullet}(b+r)$ is the complex obtained by taking global sections in the complex

$$\tilde{\mathcal{B}}^{-x,\bullet}(b+r) = (\bigwedge^x V \otimes \bigwedge^{n'+2-b'+x} \Sigma^\vee_{P,\xi} \otimes \xi_E \to \ldots \to \bigwedge^x V \otimes K_P \otimes \xi_E^{b'-x+1})$$

concentrated in degrees $x,\dots,b+r$. The complex $\tilde{\mathcal{B}}^{-x,\bullet}(b+r)[x]$ (concentrated in degrees $0,\dots,b+r-x$) is a resolution of $\Omega_P^{n+r+1-b+x}(\log \tilde{X}_t)$. Hence

$$\begin{aligned}\tilde{E}_1^{-x,y+r}(b+r) &\cong \mathbb{H}^{y+r-x}(\tilde{\mathcal{B}}^{-x,\bullet}(b+r)[x]) \\ &= \mathbb{H}^{y+r}(\tilde{\mathcal{B}}^{-x,\bullet}(b+r)).\end{aligned}$$

The spectral sequence of hypercohomology associated to the filtration bête on the complex $\tilde{\mathcal{B}}^{-x,\bullet}(b+r)[x]$ shows that condition (i) is satisfied if

$$H^{y+r-x-i-j}(P,\bigwedge^{n'+2-b'+x+i}\Sigma_{P,\xi}^{\vee}\otimes\xi_E^{i+1})=0$$

for all pairs (i,j) such that $0\le i\le y+r-x-j-1$ and $0\le j\le 1$. For every $p\ge 1$ we have an exact sequence

$$0\to\Omega_P^p\to\bigwedge^p\Sigma_{P,\xi}^{\vee}\to\Omega_P^{p-1}\to 0. \tag{15}$$

Hence it suffices to show that

$$H^{y+r-x-i-j}(P,\Omega_P^{n'+2-b'+x+i-z}\otimes\xi_E^{i+1})=0$$

if $0\le z\le 1$. Using Lemma 3.10 (i) we reduce to the condition

$$H^{y+r-x-i-j+t}(Y,\Omega_Y^u\otimes\bigwedge^v E\otimes S^w E)=0$$

where

(a) $y+r-x-i-j+t\ge 1$;

(b) $0\le u\le n+r+2-b+x+i-z$;

(c) $u+v=n+r+3-b+x+i-z+t$;

(d) $v+w=i+1$.

We can rewrite the above condition in the form

$$\oplus_{I,J}H^{y+r-x-i-j+t}(Y,\Omega_Y^u\otimes\mathcal{O}_Y(d^{<I>}+d^{(J)})=0 \tag{16}$$

for all multi–indices I and J such that $|I|=v$ and $|J|=w$. As $m_u\le m_Y+u+1$, it follows from Lemma 3.12 that it suffices to show that

$$d^{<I>}+d^{(J)}+y+r-x-i-j+t\ge m_Y+u+1.$$

By the Kodaira–Nakano vanishing theorem condition (16) is satisfied if $y+r-x-i-j+t+u>n+r+1$. Hence we may assume that $u\le n+x-y+i+j-t+1$. It follows from (c) that $v\ge y+r-b-j+2t+2-z$. To obtain the strongest possible condition, we choose $j=1$, $t=0$ and $z=1$ to obtain the minimal value of v: $v = y + r - b$. As $b + 1 - k \leq y \leq b$, we can write $y = b - s$ with $0 \leq s \leq k-1$ and $v = r - s$. From (d) we obtain that $i = r - s + w - 1$, hence $u = \dim\, Y + x - b + w$. By (a) we have $y + r - x - i - j + t = b - x - w \geq 1$. In order to minimise v we choose the maximal possible value of u: we take $w = b - x - 1$ and $u = \dim\, Y - 1$. We then choose the minimal possible value of w: $w = 0$ (and $b - x = 1$). It follows that $i = r - s - 1$, hence $s \leq \min(k-1, r-1) \leq \min(c-1, r-1)$. As we have ordered the degrees in such a way that $d_0 \geq \ldots \geq d_r$, the strongest possible condition that we obtain is

$$\sum_{\nu=\min(c,r)}^{r} d_\nu \geq m_Y + n + r.$$

(ii): From (10), applied to the pair (P, ξ) with (n, b) replaced by (n', b'), we obtain

$$''\tilde{E}_1^{x,-y+r}(b+r) \cong \mathcal{K}_{y-r,x-y+r+1}(\bigwedge^{n'+2-b'+x} \Sigma_{P,\xi}^\vee, \xi_E).$$

By (5) it suffices to show that

$$H^1(P, \bigwedge^{y-r+1} M_\xi \otimes \bigwedge^{n+r+2-b+x} \Sigma_{P,\xi}^\vee \otimes \xi_E^{x-y+r}) = 0.$$

Using the exact sequence (15) we reduce to the statement

$$H^1(P, \bigwedge^{y-r+1} M_\xi \otimes \Omega_P^{n+r+2-b+x-z} \otimes \xi_E^{x-y+r}) = 0$$

where $0 \leq z \leq 1$. Using Lemma 3.10 (ii) we reduce to the condition

$$H^{s+t+1}(Y, \bigwedge^e M_E \otimes \bigwedge^f E \otimes \Omega_Y^u \otimes \bigwedge^v E \otimes S^w E) = 0$$

for all (e, f, s, t, u, v, w) such that $s \geq 0$, $t \geq 0$ and

(a) $e + f = y - r + s + 2$;

(b) $0 \leq e \leq y - r + 1$;

(c) $u + v = n + r + 3 - b + x - z + t$;

(d) $v + w = x - y + r - s - 1$.

We can rewrite the above condition in the form

$$\oplus_{I,J,K} H^{s+t+1}(Y, \bigwedge^e M_E \otimes \Omega_Y^u \otimes \mathcal{O}_Y(d^{<I>} + d^{<J>} + d^{(K)}) = 0$$

where I, J and K are multi–indices such that $|I| = f$, $|J| = v$ and $|K| = w$. It follows from Lemma 3.12 that it suffices to show that

$$d^{<I>} + d^{<J>} + d^{(K)} + s + t + 1 \geq e + m_Y + u + 1.$$

To obtain the strongest possible condition of this type, we minimise v, f and w and choose the maximal possible values of e and u. To obtain the minimal value of f, we choose $e = y - r + 1$ in (b), hence $f = s + 1$ by (a). We take $s = 0$, $f = 1$. Define

$$i = b + r - x.$$

As $b - k + 1 \leq x - y \leq b$ we have $i - k + 1 \leq -y + r \leq i$. To obtain the minimal possible value of v, we choose $u = \dim Y$ in (c). We then take $z = 1$ and $j = 0$ to obtain $v = r - i + 1$. It follows from (d) that $w = x - y + i - 2 \geq b - k + i - 1$. Choose $w = b - k + i - 1$. We have

$$i + y \leq k + r - 1.$$

As ${}''\tilde{E}_1^{x,-y+r}(b + r) = 0$ if $y < r$ we may assume that $y \geq r$. Hence $i \leq k - 1 \leq c - 1$. We have $e = y - r + 1 \leq k - i \leq c - i$. The strongest possible condition is

$$\sum_{\nu=i}^{r} d_\nu + (b_k - k + i)d_r \geq m_Y + n + r + 1 + c - i.$$

As $b_k - k$ is constant, we can replace $b_k - k$ by $b_c - c = \mu - c$. $\square$

Remark 3.14.

(i) The expression for (C) differs from the one given in [Na2, Thm. 3]. I do not know how to obtain the bound announced in that paper. The bounds (C_i) can be replaced by the less precise bound

$$(D) \qquad (\mu - c + r + 1)d_r \geq m_Y + \dim Y + c.$$

(ii) If $n - 1 \leq c \leq n$ and $r = 0$, the bounds of Theorem 3.13 coincide with Paranjape's bound in Theorem 2.4; in the other cases we obtain more precise bounds. In the next section we shall present some examples where the condition (C) can be dropped.

(iii) In the case $Y = \mathbb{P}^{n+r+1}$, a similar result has been obtained by Asakura and Saito [AS]; they use a different version of the Jacobi ring.

(iv) If one replaces the number $\mu = [\frac{n+c}{2}]$ by an arbitrary natural number μ such that $c \leq \mu \leq n + c$ in the conditions (C_i) of Theorem 3.13, the same method of proof shows that $F^{\mu-c+k}H^{n+k}(Y_T, X_T) = 0$ for $0 \leq k \leq c$ (one replaces the numbers b_k by $\mu - c + k$ throughout the paper). Although the vanishing of $F^\mu H^{n+c}(Y_T, X_T)$ does not imply the vanishing of $H^{n+c}(Y_T, X_T)$ if $\mu > [\frac{n+c}{2}]$, this type of result can still be useful; see Section 4, Example 1 (d).

4 Applications

We apply the results of the previous section to compute degree bounds for Theorem 2.1 in a number of examples and present effective versions of some results on the cycle class, Abel–Jacobi and regulator maps for complete intersections that follow from Nori's theorem.

Suppose that $(Y, \mathcal{O}_Y(1))$ is a smooth polarised variety that satisfies the following condition:

$$H^i(Y, \Omega_Y^j(k)) = 0 \quad \text{for all } i > 0, k > 0 \text{ and } j \geq 0. \tag{17}$$

In this case the proof of Theorem 3.13 shows that condition (i) of Lemma 3.9 is satisfied; hence condition (C) of Theorem 3.13 can be omitted.

Let $(Y, \mathcal{O}_Y(1))$ be a smooth polarised variety of dimension $n+r+1$, and let $X = V(d_0, \ldots, d_r) \cap Y$ be a smooth complete intersection of dimension n, $i : X \to Y$ the inclusion map. Set $E = \mathcal{O}_Y(d_0) \oplus \ldots \oplus \mathcal{O}_Y(d_r)$ and let $g : T \to U = \mathbb{P}H^0(Y, E) \setminus \Delta$ be a smooth morphism.

It is known that Nori's theorem can be used to study regulator maps defined on Bloch's higher Chow groups; cf. [V2, Thm. 1.6].

Lemma 4.1. *If the restriction map on Deligne–Beilinson cohomology groups*

$$H_{\mathcal{D}}^{2p-k}(Y_T, \mathbb{Q}(p)) \to H_{\mathcal{D}}^{2p-k}(X_T, \mathbb{Q}(p))$$

is surjective, the image of the (rational) regulator map

$$c_{p,k} : \mathrm{CH}^p(X_t, k)_{\mathbb{Q}} \to H_{\mathcal{D}}^{2p-k}(X_t, \mathbb{Q}(p))$$

is contained in the image of $i^* : H_{\mathcal{D}}^{2p-k}(Y, \mathbb{Q}(p)) \to H_{\mathcal{D}}^{2p-k}(X_t, \mathbb{Q}(p))$ *if* $t \in T$ *is very general.*

Proof: One argues as in [Mul2, Thm. 6.2]; see also [GM1]. □

Corollary 4.2. *Put* $c = 2p - k - n + 1$. *If* $c \leq n$, *if condition (17) is satisfied and if the conditions* (C_i) *of Theorem 3.13 are satisfied for* $i = 0, \ldots, \min(c-1, r)$ *then the image of* $c_{p,k}$ *is contained in* $i^* H_{\mathcal{D}}^{2p-k}(Y, \mathbb{Q}(p))$ *if* $t \in T$ *is very general.*

Proof: The exact sequence

$$\begin{aligned} \ldots \to F^p H^{2p-k}(Y_T, X_T) \oplus H^{2p-k}(Y_T, X_T, \mathbb{Q}) \to \\ \to H_{\mathcal{D}}^{2p-k+1}(Y_T, X_T, \mathbb{Q}(p)) \to H^{2p-k+1}(Y_T, X_T) \to \ldots \end{aligned}$$

shows that $H_{\mathcal{D}}^{2p-k+1}(Y_T, X_T, \mathbb{Q}(p)) = 0$ if

$$H^{2p-k}(Y_T, X_T) = H^{2p-k+1}(Y_T, X_T) = 0.$$

Hence the statement follows from Theorem 3.13 and Lemma 4.1. □

Remark 4.3. Let $J^p_{\max}(Y)$ be the intermediate Jacobian associated to the maximal $\mathbb{Q}$–sub Hodge structure of level one in $H^{2p-1}(Y)$. In [GM1], Green and Müller–Stach prove a stronger version of Lemma 4.1 for $k = 0$: they show that the projection of the image of the Deligne cycle class map on $\mathrm{CH}^p(X)_{\mathbb{Q}}$ to the quotient $H^{2p}_{\mathcal{D}}(X, \mathbb{Q}(p))/i^* J^p_{\max}(Y)$ coincides with the image of the composed map

$$\mathrm{CH}^p(Y)_{\mathbb{Q}} \to H^{2p}_{\mathcal{D}}(Y, \mathbb{Q}(p)) \to H^{2p}_{\mathcal{D}}(X, \mathbb{Q}(p))/i^* J^p_{\max}(Y).$$

Examples:

(1) $Y = \mathbb{P}^{n+r+1}$. In this case the condition (17) is satisfied by the Bott vanishing theorem.

(a): $n = 2m$, $k = 0$, $p = m$, $c = 1$. Corollary 4.2 reduces to the Noether–Lefschetz theorem; cf. [D], [Sh]. As $H^{2m}(X, \mathbb{Z})$ contains no torsion, this result even holds with integer coefficients. The method of proof gives a more precise statement, the infinitesimal Noether–Lefschetz theorem; cf. [CGGH].

(b): $n = 2m - 1$, $k = 0$, $p = m$, $c = 2$. From Corollary 4.2 we deduce the following result: if $X = V(d_0, \ldots, d_r) \subset \mathbb{P}^{2m+r}$ is a very general smooth complete intersection of dimension $2m - 1$ ($m \geq 2$, $d_0 \geq \ldots \geq d_r$) and if

(C_0) $\sum_{i=0}^r d_i + (m-2)d_r \geq 2m + r + 2$

(C_1) $\sum_{i=1}^r d_i + (m-1)d_r \geq 2m + r + 1$

then the image of the Abel–Jacobi map

$$\psi_X : \mathrm{CH}^m_{\mathrm{hom}}(X) \to J^m(X)$$

is contained in the torsion points of $J^m(X)$. For hypersurfaces this result was proved by Green and Voisin; see [G4]. The extension to complete intersections can be found in [Na1].

(c): $n = 2m$, $k = 1$, $p = m + 1$, $c = 2$. From Corollary 4.2 we obtain the following result:

Theorem 4.4. *Let $X = V(d_0, \ldots, d_r) \subset \mathbb{P}^{2m+r+1}$ be a smooth complete intersection of dimension $2m$ ($m \geq 1$, $d_0 \geq \ldots \geq d_r$), $i : X \to \mathbb{P}^{2m+r+1}$ the inclusion map. If X is very general and if*

(C_0) $\sum_{i=0}^r d_i + (m-1)d_r \geq 2m + r + 3$

(C_1) $\sum_{i=1}^r d_i + m\, d_r \geq 2m + r + 2$

the image of the (rational) regulator map

$$c_{m+1,1} : \mathrm{CH}^{m+1}(X,1)_{\mathbb{Q}} \to H^{2m+1}_{\mathcal{D}}(X,\mathbb{Q}(m+1))$$

coincides with the image of the composed map ($N = 2m + r + 1$)

$$\mathrm{CH}^{m+1}(\mathbb{P}^N,1)_{\mathbb{Q}} \xrightarrow{\sim} H^{2m+1}_{\mathcal{D}}(\mathbb{P}^N,\mathbb{Q}(m+1)) \to H^{2m+1}_{\mathcal{D}}(X,\mathbb{Q}(m+1)).$$

As $\mathrm{CH}^{m+1}(\mathbb{P}^N,1) \cong H^{2m+1}_{\mathcal{D}}(\mathbb{P}^N,\mathbb{Z}(m+1)) \cong \mathbb{C}^*$ it follows that every element $z \in \mathrm{CH}^{m+1}(X,1)$ is regulator decomposable up to torsion (cf. [C2, p. 391] for the definition). The exceptional cases include quartic surfaces and cubic fourfolds. For these cases the regulator map has been studied in [Mul2] and [C2]; in both cases the group of regulator indecomposable higher Chow cycles is non torsion, and not even finitely generated.

(d): $n = 2m-1$, $k = 2$, $p = m+1$, $c = 2$. If we apply Corollary 4.2 to this case we find the same degree bounds as in example 1 (b). The following Lemma shows that it is possible to improve these bounds (note that the first condition of the Lemma does not imply that $H^{2m+1}(Y_T, X_T) = 0$):

Lemma 4.5. *Suppose that*

(i) $F^{m+1}H^{2m+1}(Y_T, X_T) = 0$*;*

(ii) $F^m H^{2m}(Y_T, X_T) = 0$.

Then the restriction map $i^* : H^{2m}_{\mathcal{D}}(Y_T,\mathbb{Q}(m+1)) \to H^{2m}_{\mathcal{D}}(X_T,\mathbb{Q}(m+1))$ *is surjective.*

Proof: Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\frac{H^{2m-1}(Y_T)}{F^{m+1}H^{2m-1}(Y_T)+H^{2m-1}(Y_T,\mathbb{Q})} \arrow[r, "r_1"] \arrow[d] & \frac{H^{2m-1}(X_T)}{F^{m+1}H^{2m-1}(X_T)+H^{2m-1}(X_T,\mathbb{Q})} \arrow[d] \\
H^{2m}_{\mathcal{D}}(Y_T,\mathbb{Q}(m+1)) \arrow[r, "i^*"] \arrow[d] & H^{2m}_{\mathcal{D}}(X_T,\mathbb{Q}(m+1)) \arrow[d] \\
F^{m+1}H^{2m}(Y_T) \cap H^{2m}(Y_T,\mathbb{Q}) \arrow[r, "r_2"] \arrow[d] & F^{m+1}H^{2m}(X_T) \cap H^{2m}(X_T,\mathbb{Q}) \arrow[d] \\
0 & 0.
\end{tikzcd}$$

By (ii) we have $H^{2m}(Y_T, X_T) = 0$, hence the restriction map $H^{2m}(Y_T) \to H^{2m}(X_T)$ is injective; its cokernel C carries an induced MHS. By strictness of the Hodge filtration we have an exact sequence

$$0 \to F^{m+1}H^{2m}(Y_T) \to F^{m+1}H^{2m}(X_T) \to F^{m+1}C \to 0.$$

By (i) it follows that $F^{m+1}C = 0$, hence $F^{m+1}C \cap C_{\mathbb{Q}} = 0$ and the map r_2 is surjective. It follows from (ii) that $H^{2m-1}(Y_T) \to H^{2m-1}(X_T)$ is surjective, hence the map r_1 is also surjective, and we obtain that i^* is surjective. $\square$

Theorem 4.6. *Let $X = V(d_0, \ldots, d_r) \subset \mathbb{P}^{2m+r}$ be a smooth complete intersection of dimension $2m-1$ ($m \geq 1$, $d_0 \geq \ldots \geq d_r$), $i : X \to \mathbb{P}^{2m+r}$ the inclusion map. If X is very general and if*

(C_0) $\sum_{i=0}^{r} d_i + (m-1)d_r \geq 2m + r + 2$

(C_1) $\sum_{i=1}^{r} d_i + m\, d_r \geq 2m + r + 1$

the image of the (rational) regulator map

$$c_{m+1,2} : \mathrm{CH}^{m+1}(X, 2)_{\mathbb{Q}} \to H^{2m}_{\mathcal{D}}(X, \mathbb{Q}(m+1))$$

is zero.

Proof: Using Lemmas 4.1 and 4.5 and Remark 3.14 (iv) we find that the image of $c_{m+1,2}$ is contained in the image of

$$i^* : H^{2m}_{\mathcal{D}}(\mathbb{P}^{2m+r}, \mathbb{Q}(m+1)) \to H^{2m}_{\mathcal{D}}(X, \mathbb{Q}(m+1)).$$

As $H^{2m}_{\mathcal{D}}(\mathbb{P}^{2m+r}, \mathbb{Q}(m+1)) = 0$, the image of $c_{m+1,2}$ is zero. $\square$

Remark 4.7. In [C1, (7.14)] it is shown that the image of the regulator map

$$\mathrm{CH}^2(C, 2) \to H^2_{\mathcal{D}}(C, \mathbb{Z}(2)) \cong H^1(C, \mathbb{C}^*)$$

is torsion for a very general smooth plane curve C of degree $d \geq 4$; this corresponds to the case $m = 1$, $r = 0$.

(e): $r = 0$, $c = n$. Let X_U be the universal family of smooth hypersurfaces of degree d_0 in $\mathbb{P}^{n+1}$. It follows from Theorem 3.13 that $H^{2n}(\mathbb{P}^{n+1}_U, X_U) = 0$ if $d_0 \geq 2n+1$. Voisin [V3] has constructed a higher Chow cycle cycle $Z_U \in \mathrm{CH}^n(X_U, 1)$ for the universal family X_U of hypersurfaces of degree $2n$ such that its image under the cycle class map

$$\mathrm{CH}^n(X_U, 1)_{\mathbb{Q}} \to H^{2n-1}(X_U, \mathbb{Q})$$

is not in the image of the restriction map

$$H^{2n-1}(\mathbb{P}^{n+1}_U, \mathbb{Q}) \to H^{2n-1}(X_U, \mathbb{Q}).$$

Hence $H^{2n}(\mathbb{P}^{n+1}_U, X_U) \neq 0$ if $d_0 = 2n$; this shows that the bound $d_0 \geq 2n+1$ is optimal.

(2) Let $(Y, \mathcal{O}_Y(1))$ be a polarised abelian variety. As the tangent bundle T_Y is trivial, condition (17) is satisfied by the Kodaira vanishing theorem. As Ω^m_Y is trivial for all m, the proof of Theorem 3.13 shows that we can substract $m_Y + \dim\, Y + 1$ on the right hand side of the inequality of condition (C_i). Hence condition (C_i) can be replaced by the weaker condition

(C'_i) $\sum_{\nu=i}^{r} d_\nu + (\mu - c + i)d_r \geq c - i - 1.$

This condition is empty if $c \leq 2$. For $c = 1$ we obtain a Noether–Lefschetz type result for complete intersections in abelian varieties that can be proved directly using monodromy arguments.

To state the result that is obtained for $c = 2$, we introduce some notation. Let Y be a polarised abelian variety, and let X be a smooth complete intersection in Y of dimension $2m-1$. Let $\mathrm{Hdg}^m_{\mathrm{pr}}(Y)_{\mathbb{Q}}$ be the group of primitive Hodge classes of type (m,m) on Y, and let $J^m_{\mathrm{var}}(X)$ be the intermediate Jacobian associated to $H^{2m-1}_{\mathrm{var}}(X)$. From the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & J^m(Y)_{\mathbb{Q}} & \to & H^{2m}_{\mathcal{D}}(Y, \mathbb{Q}(m)) & \to & \mathrm{Hdg}^m(Y)_{\mathbb{Q}} & \to & 0 \\ & & \downarrow i^* & & \downarrow i^* & & \downarrow i^* & & \\ 0 & \to & J^m(X)_{\mathbb{Q}} & \to & H^{2m}_{\mathcal{D}}(X, \mathbb{Q}(m)) & \to & \mathrm{Hdg}^m(X)_{\mathbb{Q}} & \to & 0 \end{array}$$

we obtain a map

$$\psi : \mathrm{Hdg}^m_{\mathrm{pr}}(Y)_{\mathbb{Q}} \to J^m_{\mathrm{var}}(X)_{\mathbb{Q}}$$

by lifting a primitive Hodge class on Y to $H^{2m}_{\mathcal{D}}(Y, \mathbb{Q}(m))$ and restricting to X. By passage to the quotient, the Abel–Jacobi map ψ^m_X induces a map

$$\overline{\psi^m_X} : \mathrm{CH}^m_{\mathrm{hom}}(X)_{\mathbb{Q}} \to J^m_{\mathrm{var}}(X)_{\mathbb{Q}}.$$

Theorem 4.8. *Let $(Y, \mathcal{O}_Y(1))$ be a polarised abelian variety and let X be a smooth complete intersection in Y of dimension $2m-1$. If X is very general, the image of $\overline{\psi^m_X}$ is contained in the image of the map*

$$\psi : \mathrm{Hdg}^m_{\mathrm{pr}}(Y)_{\mathbb{Q}} \to J^m_{\mathrm{var}}(X)_{\mathbb{Q}}.$$

Proof: This follows from Corollary 4.2 if we take $c = 2$, $k = 0$, $p = m$ and use the improved conditions (C'_i). $\square$

(3) The condition (17) is also satisfied if $(Y, \mathcal{O}_Y(1))$ is a smooth polarised toric variety, by the Bott–Danilov–Steenbrink vanishing theorem; see [BC, Thm. 7.1].

(4) We consider an example mentioned in the introduction of Nori's paper. Let $Y \subset \mathbb{P}^7$ be a smooth quadric, and let $X = Y \cap V(d_0, d_1)$, $d_0 \geq d_1$, be a smooth complete intersection in Y. In this case condition (17) is not satisfied. As $m_Y = 1$ [Pa], Paranjape's results show that $H^{n+k}(Y_T, X_T) = 0$ for $k \leq 3$ if $d_1 \geq 9$. As $c = 3$ and $r = 1$, the conditions of Theorem 3.13 read $(C) : d_1 \geq 6$, $(C_0) : d_0 + d_1 \geq 10$, $(C_1) : 2d_1 \geq 9$. Using precise vanishing theorems for the groups $H^i(Y, \Omega^j_Y(k))$ (cf. [Sn]), we find that the bound in condition (C) can be improved to $d_1 \geq 5$.

Let $\mathrm{Griff}^3(X)$ be the Griffiths group of codimension 3 cycles on X. Using [No, Thm. 1] it follows that $\mathrm{Griff}^3(X) \otimes \mathbb{Q} \neq 0$ if X is very general and $d_1 \geq 5$. Bloch and Srinivas have proved that $\mathrm{Griff}^3(X) \otimes \mathbb{Q} = 0$ if X is a smooth Fano fourfold; see [BS, Thm. 2 (i)]. Hence $\mathrm{Griff}^3(X) \otimes \mathbb{Q} = 0$ if $d_0 + d_1 < 6$.

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