Effective bounds for Nori's connectivity theorem

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Abstract. We announce an effective version of Nori's connectivity theorem.Uisng this theorem, we obtain effective versions of results of Green, Voisin and Müller–Stach on the image of the Abel–Jacobi and regulator maps for very general complete intersections.

Résumé. On annonce une version effective du théorème de connectivité de Nori. À l'aide de ce théorème on obtient des versions effectives des résultats de Green, de Voisin et de Müller–Stach sur l'image de l'application d'Abel–Jacobi et de l'application régulateur pour les intersections complètes très générales.

Version française abrégée. Soit $(Y, \mathcal{O}_Y(1))$ une variété polarisée complexe et lisse de dimension $n+r+1$, et soit $E = \bigoplus_{i=0}^{r} \mathcal{O}_Y(d_i)$ $(d_i > 0,\ i = 0, \ldots, r)$. On pose $S = \prod_{i=0}^{r} \mathbb{P}H^0(Y, \mathcal{O}_Y(d_i))$. Soit $X_S \subset Y \times S$ la famille universelle des intersections complètes de multidegré $(d_0, \ldots, d_r)$ dans Y, et soit $g : T \to S$ un morphisme lisse. On définit $X_T = X_S \times_S T$, $Y_T = Y \times T$.

Théor'eme 1 (Nori [12], Thm. 4.) Pour tout nombre entier positif $c \leq n$, il exist un nombre entier positif $N = N(Y, \mathcal{O}_Y(1), c)$ tel que $H^{n+k}(Y_T, X_T, \mathbb{Q}) = 0$ pour $k \leq c$ si $_m in(d_0, \ldots, d_r) \geq N$.

Dans [13], K. Paranjape a démontré une version effective du theéorème de Nori en utilisant la régularité de Castelnuovo–Mumford (voir aussi [2] et [14]). Soit m_i la régularité de Ω^i_Y. On pose $m_Y = \max\{m_i - i - 1 \mid 0 \leq i \leq \dim Y\}$.

Théorème 2 (Paranjape) Avec les notations du théorème 1, on a:

$$N(Y, \mathcal{O}_Y(1), c) \leq m_Y + n + c + 1.$$

Dans cette note on impose une condition un peu plus forte sur le changement de base, qui suffit pour déduire les conséquences géométriuqes du théorème de Nori; on obtient des bornes sur les degrés qui sont plus précises que celles du théorème 2. Les détails seront publiés dans [11]. Soit $U \subset \mathbb{P}H^0(Y, E)$ l'ouvert qui paramètre les intersections complètes lisses. Pour un nombre entier positif $c \leq n$, on définit:

$$b_c = \left[\frac{n+c}{2}\right],\ b_{c-i} = b_c - i,\ 1 \leq i \leq c-1. \tag{1}$$

Théorème 3. Soit $T \to U$ un changement de base lisse , soit $c \leq n$ un entier positif et soient $b_1, \ldots b_c$ définis comme dans (1). On suppose que $d_0 \geq \ldots \geq d_r$. Si les conditions suivantes sont vérifiées:

(C) $d_r \geq m_Y + n + 2(c - b_c) + 2$;

(C_i) $\sum_{\nu=i}^{r} d_\nu = (b_c - c + i)d_r \geq m_Y + \dim Y + c - i$ pour $0 \leq i \leq \min(c-1, r)$

alors $H^{n+k}(Y_T, X_T, \mathbb{Q}) = 0$ pour tout entier $k \leq c$.

Les conditions (C_i) peuvent être remplacées par la condition

(D) $(b_c - c + r + 1)d_r \geq m_Y + n + c + r + 1$.

La condition (D) équivaut à celle de Paranjape si $r = 0$ et $n - 1 \leq c \leq n$; dans les autres cas, elle est plus précise. Si $(Y, \mathcal{O}_Y(1))$ satisfait la condition

$$H^i(Y, \Omega_Y^j(k)) = 0 \quad \text{pour tous les entiers } i > 0, j \geq 0, k > 0 \tag{2}$$

on peut omettre la condition (C). Soit $J^p_{\max}(Y) \subset J^p(Y)$la sous–variété abélienne associée à la sous–structure de Hodge maximale contenue dans $F^{p-1}H^{2p-1}(Y) \cap H^{2p-1}(Y, \mathbb{Q})$, et soit $\pi : H^{2p}_{\mathcal{D}}(X, \mathbb{Q}(p)) \to H^{2p}_{\mathcal{D}}(X, \mathbb{Q}(p))/i^* J^p_{\max}(Y)$ la projection.

Théorème 4. Soit $(Y, \mathcal{O}_Y(1))$ une variété poalrisée, soit $X \subset Y$ une intersection complète de multidegré $(d_0, \ldots, d_r)$ $(d_0 \geq \ldots \geq d_r)$ dans Y, et soit $i : X \to Y$ l'inclusion. Soit $c \leq n$ un entier positif. Si X est très générale et si $(Y, \mathcal{O}_Y(1))$ vérifie les conditions (2) et (C_i), $0 \leq i \leq \min(c-1, r)$, alors on a:

(i) L'image de l'application $\pi \circ_c l_{\mathcal{D},X} : \mathrm{CH}^p(X)_{\mathbb{Q}} \to H^{2p}_{\mathcal{D}}(X, \mathbb{Q}(p))/i^* J^p_{\max}(Y)$ coïncide avec l'image de l'application $\pi \circ i^* \circ \mathrm{cl}_{\mathcal{D},Y} : \mathrm{CH}^p(Y)_{\mathbb{Q}} \to H^{2p}_{\mathcal{D}}(X, \mathbb{Q}(p))/i^* J^p_{\max}(Y)$ si $2p \leq n + c - 1$;

(ii) L'image de l'application régulateur $c_{p,m} : \mathrm{CH}^p(X, m)_{\mathbb{Q}} \to H^{2p-m}_{\mathcal{D}}(X, \mathbb{Q}(p))$ est contenu dans l'image de $i^* : H^{2p-m}_{\mathcal{D}}(Y, \mathbb{Q}(p)) \to H^{2P-m}_{\mathcal{D}}(X, \mathbb{Q}(p))$ si $2p - m \leq n + c - 1$.

Let $(Y, \mathcal{O}_Y(1))$ be a smooth polarized complrc variety of dimension $n + r + 1$, and set $E = \bigoplus_{i=0}^{r} \mathcal{O}_Y(d_i)$ $(d_i > 0,\ i = 0, \ldots, r)$. Set $S = \prod_{i=0}^{r} \mathbb{P}H^0(Y, \mathcal{O}_Y(d_i))$, and let $X_S \subset Y \times S$ be the universal family of complete intersections of multidegree $(d_0, \ldots, d_r)$ in Y. Let $T \to S$ be a smooth morphism. Define $X_T = X_S \times_S T$, $Y_T = Y \times T$. In [12] M. Nori proved the following result, which can be seen as a relative version of the classical Lefschetz hyperplane theorem:

Theorem 1. (Nori [12], Thm. 4) For every natural bnumber $c \leq n$ there exists a natural number $N = N(Y, \mathcal{O}_Y(1), c)$ such that $H^{n+k}(Y_T, X_T; \mathbb{Q}) = 0$ for all $k \leq c$ if $\min(d_0, \ldots, d_r) \geq N$.

Remark. Theorem 1 remains valid if one replaces S by $\mathbb{P}H^0(Y, E)$ or by $U = \mathbb{P}H^0(Y, E) \setminus \Delta$, the complement of the discriminant locus (see [6], Lecture 8).

The proof of Theorem 1 uses the mixed Hodge structure on the relative cohomology groups $H^{n+k}(Y_T, X_T; \mathbb{C})$ constructed by Deligne [4]. As the weight filtration satisfies $\mathrm{Gr}_i^W H^{n+k}(Y_T, X_T) = 0$ for all $i \leq n+k-1$, it follows that $H^{n+k}(Y_T, X_T) = 0$ if $F^{b_k} H^{n+k}(Y_T, X_T) = 0$ for some integer $0 \leq b_k \leq [\frac{n+k}{2}]$. In his paper, Nori chooses $b_k = k$ for all $k \leq c$; our choice is

$$b_c = \left[\frac{n+c}{2}\right], \; b_{c-i} = b_c - i, \; 1 \leq i \leq c-1. \tag{3}$$

K. Paranjape proved in [13] an effective version of Theorem 1using Castelnuovo–Mumford regularity (see also [2] and [14]). Let m_i be the regularity of the vector bundle Ω^i_Y, i.e., $m_i = \min\{k \mid H^j(Y, \Omega^i_Y(k-j)) = 0 \text{ for all } j > 0\}$, and set $m_Y =_m ax\{m_i - i - 1 \mid 0 \leq i \leq \dim Y\}$. For example, $m_Y = 0$ if Y is a projective space and $m_Y = 1$ if Y is a smooth quadric.

Theorem 2. (Paranjape) With the notation of Theorem 1, we have

$$N(Y, \mathcal{O}_Y(1), c) \leq m_Y + n + c + 1.$$

In this note, we impose a somewhat stronger condition on the base change, which suffices to deduce the geometric consequences of Nori's theorem; the degree bounds that we obtain are more precise than those of Theorem 2. The proof of this result is based on unpublished manuscripts of Green and Müller–Stach and on a remark of Nori ([12], Remark 3.10). We use Koszul cohomology computations and Jacobi modules. Details will appear in [11].

Theorem 3. Let $d_0 \geq \ldots \geq d_r$, let $U = \mathbb{P}H^0(Y, E) \setminus \Delta$ be the complement of the dsicriminant locus and let $T \to S$ be a smooth morphism. Let $c \leq n$ be a natural number, and let $b_1, \ldots, b_c$ be defined as in (3). Suppose that

(C) $d_r \geq m_Y + n + 2(c - b_c) + 2$;

(C_i) $\sum_{\nu=i}^r d_\nu = (b_c - c + i) d_r \geq m_Y + \dim Y + c - i$ pour $0 \leq i \leq \min(c-1, r)$.

Then $H^{n+k}(Y_T, X_T) = 0$ for all $k \leq c$.

Remarks.

1. Theorem 3 has been proved independently by M. Asakura and S. Saito [1] for the case $Y = \mathbb{P}^{n+r+1}$; their approach is similar to ours, but they use a different version of the Jacobi module

2. Conditions $(C_0), \ldots, (C_{c-1})$ can be replaced by the less precise condition

   (D) $(b_c - c + r + 1) d_r \geq m_Y + n + c + r + 1$.

   Condition (D) is equivalent to Paranjape's bound if $r = 0$ and $n - 1 \leq c \leq n$; in the other case it is more precise.

If $(Y, \mathcal{O}_Y(1))$ is a polarized variety that satisfies the condition

$$H^i(Y, \Omega^j_Y(k)) = 0 \text{ pour tous les entiers } i > 0, j \geq 0, k > 0 \tag{4}$$

we can omit condition (C). As a Corollary of Theorem 3, we obtain an effective version of a result of Green and Müller–Stach that describes the image of the Deligne cycle class map for complete intersections (see [7]) and its subsequent generalization to higher Chow groups (see [9], Theorem 6.2). Let $J^p_{\max}(Y) \subset J^p(Y)$ be the abelian subvariety associated to the maximal sub–Hodge structure contained in $F^{p-1}H^{2p-1}(Y, \mathbb{C}) \cap H^{2p-1}(Y, \mathbb{Q})$, and let $\pi : H^{2p}_{\mathcal{D}}(X, \mathbb{Q}(p)) \to H^{2p}_{\mathcal{D}}(X, \mathbb{Q}(p))/i^* J^p_{\max}(Y)$ be the projection map.

Theorem 4. Let $(Y, \mathcal{O}_Y(1))$ be a smooth polarized variety of dimension $n + r + 1$, and let $c \leq n$ be a natural number. Let $X = V(d_0, \ldots, d_r) \cap Y$ $(d_0 \geq \ldots \geq d_r)$ be a smooth complete intersection of dimension n in Y with inclusion map $i : X \to Y$. If X is very general and if $(Y, \mathcal{O}_Y(1))$ satisfies conditions (2) and (C_i) for all $0 \leq i \leq \min(c-1, r)$ then

(i) The image of the map $\pi \circ_c l_{\mathcal{D},X} : \mathrm{CH}^p(X)_{\mathbb{Q}} \to H^{2p}_{\mathcal{D}}(X, \mathbb{Q}(p))/i^* J^p_{\max}(Y)$ coincides with the image of the map $\pi \circ i^* \circ \mathrm{cl}_{\mathcal{D},Y} : \mathrm{CH}^p(Y)_{\mathbb{Q}} \to H^{2p}_{\mathcal{D}}(X, \mathbb{Q}(p))/i^* J^p_{\max}(Y)$ if $2p \leq n + c - 1$;

(ii) The image of the regulator map $c_{p,m} : \mathrm{CH}^p(X, m)_{\mathbb{Q}} \to H^{2p-m}_{\mathcal{D}}(X, \mathbb{Q}(p))$ is contained in the image of $i^* : H^{2p-m}_{\mathcal{D}}(Y, \mathbb{Q}(p)) \to H^{2P-m}_{\mathcal{D}}(X, \mathbb{Q}(p))$ si $2p - m \leq n + c - 1$.

Examples.

1. If $(Y, \mathcal{O}_Y(1)) = (\mathbb{P}^{n+r+1}, \mathcal{O}_{\mathbb{P}}(1))$ then condition (2) is satisfied by the Bott vanishing theorem. Taking $c = 1$ in Theorem 4, we obtain the Noether–Lefschetz theorem (see [3]). For $c = 2$ we obtain a generalization of a theorem of Green and Voisin [5] on the image of the Abel–Jacobi map to complete intersections (see also [10]).

2. Let $(Y, \mathcal{O}_Y(1))$ be a smooth polarized toric variety. Then condition (2) is satisfied by the Bott–Danilov–Steenbrink vanishing theorem.

3. Let $(Y, \mathcal{O}_Y(1))$ be a polarized abelian variety. Then condition (2) is satisfied by the Kodaira vanishing theorem. In this case, condition (C_i) can be replaced by the weaker condition

   (C'_i) $\sum_{\nu=i}^r d_\nu + (b_c - c + i)d_r \geq c - i - 1$.

   This condition is empty if $c \leq 2$.

   If Y is a homogeneous space (e.g. a Grassmann variety or a quadric) one can sometimes compute reasonably precise degree bounds using the Bott vanishing theorem. For instance, let $Y \subset \mathbb{P}^7$ be a smooth quadric. Explicit computations show that if

$X = V(d_0, d_1) \cap Y$ is very general and $d_1 = \min(d_0, d_1) \geq 5$ then the conclusion of Theorem 3 holds for $c = 3$. Using a theorem of Nori ([12], Theorem 1) we find that the Griffiths group $\mathrm{Griff}^3(X) \otimes \mathbb{Q}$ is nonzero if $d_1 \geq 5$.

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