Relative Chow-Künneth decompositions for conic bundles and Prym varieties

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Abstract. We construct a relative Chow-Künneth decomposition for a conic bundle over a surface such that the middle projector gives the Prym variety of the associated double covering of the discriminant of the conic bundle. This gives a refinement (up to an isogeny) of Beauville's theorem on the relation between the intermediate Jacobian of the conic bundle and the Prym variety of the double covering.

Introduction

Let $f : X \to S$ be a conic bundle over a surface, i.e., X is a smooth projective threefold over k, S is a projective surface over k and the fibers of f are conics, where k is a perfect field of characteristic different from 2. Let C be the discriminant of f; it is a curve whose singularities are ordinary double points, see [3]. (Here C is not necessarily connected.) The singularities of C are the points $s \in S$ such that $f^{-1}(s)$ is a double line. Put $X_C = f^{-1}(C)$, and let $\widetilde{X_C}$ be its normalization (which is smooth). Let $\widetilde{C}$ denote $F_1(X_C/C)$ the relative Fano scheme of lines of X_C over C (i.e. its fiber over $s \in C$ consists of the irreducible components of $f^{-1}(s)$). In [3, 0.3] it is shown that the morphism $\widetilde{C} \to C$ is an admissible double covering, i.e., it is an étale double covering outside $\mathrm{Sing}\,(C)$ and the inverse image of a double point of C is a double point of $\widetilde{C}$. Let D and C' denote respectively the normalizations of $\widetilde{C}$ and C (which are denoted respectively by $\widetilde{N}$ and N in [3], [6]). Let C_j be the irreducible components of C. Let C'_j be the normalization of C_j, and D_j be the union of the irreducible components of D whose image in C is C_j. Suppose that each D_j is irreducible, i.e. the restriction of the double covering $\widetilde{C} \to C$ over $C_j \setminus \mathrm{Sing}\, C$ is nontrivial. If k is not algebraically closed, we assume this after taking the base change $k \to \bar{k}$.

Let $\mathcal{P}_X$ be the generalized Prym variety associated to the double covering $\widetilde{C} \to C$, as defined in [3, 0.3.2]. Then $\mathcal{P}_X$ is isogenous to the product of the Prym varieties of D_j/C'_j, see [3], Prop. 0.3.3 (cf. also [6, Prop. 1.5]). Let σ_j be the involution of D_j associated to the double covering

$$\rho_j : D_j \to C'_j.$$

This gives an idempotent

$$\widetilde{\pi}_j := (id - \sigma_j)/2 \in \mathrm{Cor}_k^0(D_j, D_j) = \mathrm{CH}^1(D_j \times_k D_j)_{\mathbf{Q}},$$

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where σ_j is identified with its graph. We define a Chow motive, called the Prym motive, by

$$\mathrm{Prym}(D_j/C'_j) := (D_j, \widetilde{\pi}_j).$$

It is identified with the Prym variety of D_j/C'_j by Weil's theory of correspondences between curves. Let $h^i(X)$, $h^i(S)$ denote the i-th component of the Chow-Künneth decomposition, where the existence of $h^i(S)$ was proved in [17] (see also [18]). Let ℓ be a prime different from the characteristic of k, and let $\mathrm{CH}^p_{\mathrm{alg}}(X)_{\mathbf{Q}}$ be the subgroup of $\mathrm{CH}^p(X)_{\mathbf{Q}}$ consisting of cycles algebraically equivalent to zero. The following gives a generalization of [3], [6], and has been conjectured by the first author [19].

Theorem 1. *There is a self-dual Chow-Künneth decomposition for X together with the isomorphisms of Chow motives*

$$h^i(X) \cong \begin{cases} h^3(S) \oplus h^1(S)(-1) \oplus (\bigoplus_j \mathrm{Prym}(D_j/C'_j)(-1)) & \text{if } i = 3, \\ h^i(S) \oplus h^{i-2}(S)(-1) & \text{if } i \neq 3, \end{cases}$$

where (-1) denotes the Tate twist of Chow motives. In particular, if $H^1(S_{\bar{k}}, \mathbf{Q}_\ell) = 0$ or equivalently $\mathrm{CH}^1_{\mathrm{alg}}(S_{\bar{k}})_{\mathbf{Q}} = 0$, then

$$h^3(X) = \bigoplus_j \mathrm{Prym}(D_j/C'_j)(-1).$$

Theorem 1 gives a refinement (up to an isogeny) of a theorem of Beauville [3] in the case of conic bundles over $\mathbf{P}^2_{\mathbf{C}}$ with smooth C, where he gave an isomorphism between the intermediate Jacobian of X and the Prym variety $\mathcal{P}_X$ of D/C as principally polarized abelian varieties over $\mathbf{C}$. Note that Theorem 1 in the case $k = \mathbf{C}$ implies an isomorphism of $\mathbf{Q}$-Hodge structures

$$H^3(X) = H^3(S) \oplus H^1(S)(-1) \oplus (\bigoplus_j \mathrm{Coker}(H^1(C'_j) \to H^1(D_j))(-1)).$$

To show Theorem 1, we consider the relative Chow-Künneth decomposition for f (see [9], [14], [15], [22]) in the 'weak' and 'strong' sense (see 1.6 for notation), and prove the following (which has been studied in [19]).

Theorem 2. *There is a unique self-dual relative Chow-Künneth decomposition for f in the weak sense, and the projectors $\pi_{f,-1}$, $\pi_{f,0}$ and $\pi_{f,1}$ define Chow motives isomorphic to (S, Δ_S), $\bigoplus_j \mathrm{Prym}(D_j/C'_j)(-1)$ and $(S, \Delta_S)(-1)$ respectively, where Δ_S is the diagonal of $S \times S$. Moreover, there is a unique self-dual relative Chow-Künneth decomposition for f in the strong sense, and the relative projector $\pi_{f,0,j}$ corresponding to the direct factor supported on C_j defines a Chow motive isomorphic to $\mathrm{Prym}(D_j/C'_j)(-1)$.*

The proof of Theorem 2 follows from a calculation of the composition of certain relative correspondences by decomposing these into the compositions of more elementary correspondences. Here we have to show the vanishing of certain 'phantom' motives. The construction of the middle projector is due to the first author [19].

From Theorem 2 we can deduce the following generalization of [3], Th. 3.6 (where $k = \bar{k}$ and $S = \mathbf{P}^2$) and [6], Th. 2.6 (where $k = \bar{k}$, $\mathrm{char}\, k = 0$ and C is irreducible).

Corollary 1. *There is a canonical isomorphism*

$$\mathrm{CH}^2_{\mathrm{alg}}(X)_{\mathbf{Q}} = \mathrm{CH}^2_{\mathrm{alg}}(S)_{\mathbf{Q}} \oplus \mathrm{CH}^1_{\mathrm{alg}}(S)_{\mathbf{Q}} \oplus \mathcal{P}_X(k)_{\mathbf{Q}}.$$

In particular, if $H^1(S_{\bar{k}}, \mathbf{Q}_\ell) = 0$ *or equivalently* $\mathrm{CH}^1_{\mathrm{alg}}(S_{\bar{k}})_{\mathbf{Q}} = 0$*, then*

$$\mathrm{CH}^2_{\mathrm{alg}}(X)_{\mathbf{Q}} = \mathrm{CH}^2_{\mathrm{alg}}(S)_{\mathbf{Q}} \oplus \mathcal{P}_X(k)_{\mathbf{Q}}.$$

If furthermore $\mathrm{CH}^2(S)_{\mathbf{Q}} = \mathbf{Q}$*, then*

$$\mathrm{CH}^2_{\mathrm{alg}}(X)_{\mathbf{Q}} = \mathcal{P}_X(k)_{\mathbf{Q}}.$$

In case $k = \bar{k}$ and $\mathrm{char}\, k = 0$, the condition $\mathrm{CH}^2(S)_{\mathbf{Q}} = \mathbf{Q}$ implies $H^i(S, \mathcal{O}_S) = 0$ for $i = 1, 2$, see [16]. Its converse was conjectured by S. Bloch [7], and has been proved at least if S is not of general type, see [8] and also [2], etc.

In Section 1 we review some basic facts related to conic bundles and Chow-Künneth decompositions. In Section 2 we prove the main theorems.

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1. Preliminaries

1.1. Conic bundles. Let $f : X \to S$ be a conic bundle with $\dim X = 3$ and $\dim S = 2$. Let C be the discriminant. It is a divisor with normal crossings, see [3]. Locally X is a subvariety of $U \times \mathbf{P}^2$ defined by a relative quadratic form where U is an open subvariety of S. Note that $X_s := f^{-1}(s)$ is a union of two lines (resp. a line) in $\mathbf{P}^2$ if s is a smooth (resp. singular) point of C. Let $X_C = f^{-1}(C)$, and let $\widetilde{X_C}$ be its normalization. Let C' be the normalization of C. Then $\widetilde{X_C}$ is smooth, and is a $\mathbf{P}^1$-bundle over a double covering D of C' (its fibers are lines in $\mathbf{P}^2$ locally).

Let C_j be the irreducible components of C. Let C'_j be the normalization of C_j, and D_j be the union of the irreducible components of D whose image in C is C_j. Put $C^o_j = C_j \setminus \mathrm{Sing}\, C$. In the sequel we shall identify C^o_j with the corresponding subset of the normalization C'_j. Let

$$\rho_j : D_j \to C'_j$$

be the double covering, and put $D^o_j = \rho_j^{-1}(C^o_j)$. We assume that the D_j are irreducible, i.e.

(1.1.1) The restriction $D^o_j \to C^o_j$ is a nontrivial double covering for all j.

In case k is not algebraically closed, we assume (1.1.1) after taking the base change $k \to \bar{k}$.

1.2. Example. Let E_j be line bundles on S, and a_j be sections of $E_j \otimes E_j$ for $j = 0, 1, 2$. Assume the zeros of a_j are smooth divisors C_j and their union C is a divisor with normal crossings on S. Then these define a conic bundle $f : X \to S$ such that X is locally defined by

$$\textstyle\sum_{0 \le j \le 2} a_j x_j^2 = 0 \quad \text{in } U \times \mathbf{P}^2,$$

trivializing E_j locally over an open subvariety U of S. The condition (1.1.1) is satisfied if $C_j \cap \operatorname{Sing} C \ne \emptyset$ for any j, see (1.3.4) below.

1.3. Decomposition theorem. With the notation and the assumptions of (1.1) assume k is algebraically closed. Let $\iota_j : C_j^o := C_j \setminus \operatorname{Sing} C \to C_j$ denote the inclusion. By [5] there is a noncanonical isomorphism

$$\mathbf{R}f_*\mathbf{Q}_{l,X}[3] \simeq \textstyle\bigoplus_{-1 \le i \le 1} {}^pR^i f_*(\mathbf{Q}_{l,X}[3])[-i] \quad \text{in } D_c^b(S, \mathbf{Q}_l), \tag{1.3.1}$$

together with canonical isomorphisms

$$\begin{gathered} {}^pR^{-1} f_*(\mathbf{Q}_{l,X}[3]) = \mathbf{Q}_{l,S}[2], \quad {}^pR^1 f_*(\mathbf{Q}_{l,X}[3]) = \mathbf{Q}_{l,S}(-1)[2], \\ {}^pR^0 f_*(\mathbf{Q}_{l,X}[3]) = \textstyle\bigoplus_j (\iota_j)_* L_j[1]. \end{gathered} \tag{1.3.2}$$

Here L_j is the restriction to C_j^o of $((\rho_j)_*\mathbf{Q}_{\ell,D_j}/\mathbf{Q}_{\ell,C_j'})(-1)$ with $\rho_j : D_j \to C_j'$ the natural morphism. It is a smooth $\mathbf{Q}_\ell$-sheaf of rank 1. See [5] for the definition of $D_c^b(S, \mathbf{Q}_\ell)$ and ${}^pR^i f_* := {}^p\mathcal{H}^i\mathbf{R}f_*$. Note that

$$\text{Condition (1.1.1) is equivalent to } \Gamma(C_j^o, L_j) = 0 \text{ for any } j. \tag{1.3.3}$$

Since the fiber of f at $s \in C_j \setminus C_j^o$ is a line, the stalk of $(\iota_j)_*L_j$ at $s \in C_j \setminus C_j^o$ vanishes and hence $(\iota_j)_*L_j = (\iota_j)_!L_j$, i.e. the local monodromy of L_j around s is nontrivial. So we get

$$\text{Condition (1.1.1) is satisfied if } C_j \cap \operatorname{Sing} C \ne \emptyset \text{ for any } j. \tag{1.3.4}$$

Note that the last condition is equivalent to that any connected component of C has a singular point.

1.4. Chow motives. Let X, Y be smooth projective varieties over a perfect field k. Assume X is equidimensional. Then the group of correspondences is defined by

$$\operatorname{Cor}_k^i(X, Y) = \operatorname{CH}^{\dim X + i}(X \times_k Y)_{\mathbf{Q}}. \tag{1.4.1}$$

In general, we take the direct sum over the connected components of X. A *Chow motive* is defined by (X, π, i) where $\pi \in \operatorname{Cor}_k^0(X, X)$ is an idempotent (i.e. $\pi^2 = \pi$) and $i \in \mathbf{Z}$. Note that i is related to morphisms of Chow motives which are defined by

$$\operatorname{Hom}((X, \pi, i), (Y, \pi', j)) = \pi' \circ \operatorname{Cor}_k^{j-i}(X, Y) \circ \pi. \tag{1.4.2}$$

Sometimes we denote $(X, \pi, 0)$ by (X, π). The *Tate twist* of Chow motives is defined by

$$(X, \pi, i)(m) = (X, \pi, i + m). \tag{1.4.3}$$

Similarly we can define relative Chow motives (see [9], [12]) using relative correspondences defined as below.

1.5. Relative correspondences. Let X, Y be smooth varieties over a perfect field k with projective morphisms $f : X \to S$, $g : Y \to S$ over k. The group of relative correspondences is defined by

$$\mathrm{Cor}_S^i(X,Y) = \mathrm{CH}_{\dim Y - i}(X \times_S Y)_{\mathbf{Q}}, \tag{1.5.1}$$

if Y is equidimensional. In general we take the direct sum over the connected components of Y. The composition of relative correspondences is defined by using the pull-back associated to the cartesian diagram

$$\begin{array}{ccc} X \times_S Y \times_S Z & \rightarrow & (X \times_S Y) \times_k (Y \times_S Z) \\ \downarrow & & \downarrow \\ Y & \rightarrow & Y \times_k Y, \end{array}$$

together with the pushforward by $X \times_S Y \times_S Z \to X \times_S Z$, see [9], [13]. There is a natural morphism

$$\mathrm{Cor}_S^i(X,Y) \to \mathrm{Cor}_k^i(X,Y), \tag{1.5.2}$$

which is compatible with composition. This induces a forgetful functor from the category of relative Chow motives over S to the category of Chow motives over k, see [9].

If $k = \bar{k}$ we have the action of correspondences

$$\begin{aligned} \mathrm{Cor}_S^i(X,Y) &\to \mathrm{Hom}(\mathbf{R}f_*\mathbf{Q}_{l,X}, \mathbf{R}g_*\mathbf{Q}_{l,Y}(i)[2i]) \\ &\to \textstyle\bigoplus_j \mathrm{Hom}({}^pR^jf_*\mathbf{Q}_{l,X}, {}^pR^{j+2i}g_*\mathbf{Q}_{l,Y}(i)). \end{aligned} \tag{1.5.3}$$

This is compatible with the composition of correspondences, see loc. cit.

1.6. Relative Chow-Künneth decomposition. With the notation and the assumptions of (1.3), assume there are mutually orthogonal idempotents

$$\pi_{f,i} \in \mathrm{Cor}_S^0(X,X) = \mathrm{CH}^1(X \times_S X)_{\mathbf{Q}} \quad \text{for } i = -1, 0, 1,$$

such that $\sum_i \pi_{f,i} = \Delta_X$ where Δ_X denotes the diagonal. We say that they define a *relative Chow-Künneth decomposition for f in the weak sense* if the action of $\pi_{f,i}$ on ${}^pR^jf_*(\mathbf{Q}_{l,X}[3])$ is the identify for $i = j$, and vanishes otherwise, see [22]. In case k is not algebraically closed, we say that mutually orthogonal idempotents define a relative Chow-Künneth decomposition if their base changes by $k \to \bar{k}$ do.

Let $\pi_{f,i}$ be mutually orthogonal relative projectors defining a relative Chow-Künneth decomposition for f, and $\pi_{f,0,j}$ be mutually orthogonal relative projectors such that $\pi_{f,0} = \sum_j \pi_{f,0,j}$. (Note that $\pi_{f,i} \circ \pi_{f,0,j} = \pi_{f,i} \circ \pi_{f,0} \circ \pi_{f,0,j} = 0$ for $i = \pm 1$.) We say that they define a *relative Chow-Künneth decomposition for f in the strong sense* if the action of $\pi_{f,0,j}$ on the direct factor supported on $C_{j'}$ is the identify for $j = j'$, and vanishes otherwise, see [9], [14]. In case k is not algebraically closed, the above condition should be satisfied for the base change by $k \to \bar{k}$, where the direct factor supported on $C_{j'}$ should be replaced by the direct factor supported on the base change of $C_{j'}$.

We say that a decomposition is *self-dual* if the projectors satisfy the self-duality

$$\pi_{f,i} = {}^t\pi_{f,-i} \ \text{ and } \ \pi_{f,0,j} = {}^t\pi_{f,0,j} \text{ (in the strong case).}$$

1.7. Heuristic argument. With the notation and the assumptions of (1.3), assume that the decomposition (1.3.1) holds in the derived category of (conjectural) motivic sheaves $D^b\mathcal{M}(S)$ (see [4]) where the following isomorphism should hold:

$$(1.7.1) \qquad \mathrm{End}_{D^b\mathcal{M}(S)}(\mathbf{R}f_*\mathbf{Q}_X^{\mathcal{M}}[3]) = \mathrm{Cor}_S^0(X,X)\,(:= \mathrm{CH}^1(X\times_S X)_{\mathbf{Q}}).$$

Here $\mathbf{Q}_X^{\mathcal{M}} \in D^b\mathcal{M}(X)$ is the constant sheaf. (In case $k = \mathbf{C}$ we may assume $\mathcal{M}(X) = \mathrm{MHM}(X)$, see Remark (1.8) below.) Then (1.3.1) and (1.7.1) should induce a relative Chow-Künneth decomposition in the weak sense by taking the projection to each direct factor. If we have another relative Chow-Künneth decomposition in the weak sense, then the corresponding projectors $\pi_{f,i}$ are identified with endomorphisms

$$\pi_{f,i} : \mathbf{R}f_*\mathbf{Q}_X^{\mathcal{M}}[3] \to \mathbf{R}f_*\mathbf{Q}_X^{\mathcal{M}}[3],$$

and (1.3.1) gives a decomposition $\pi_{f,i} = \bigoplus_{r,s}(\pi_{f,i})_{r,s}$ such that $(\pi_{f,i})_{r,s}$ is identified with

$$(\pi_{f,i})_{r,s} \in \mathrm{Ext}^{s-r}({}^pR^sf_*(\mathbf{Q}_X^{\mathcal{M}}[3]), {}^pR^rf_*(\mathbf{Q}_X^{\mathcal{M}}[3])) \ \ (i,r,s \in \{-1,0,1\}).$$

In particular, $(\pi_{f,i})_{r,s} = 0$ for $r > s$. We have also

$$(\pi_{f,i})_{i,i} = id, \text{ and } (\pi_{f,i})_{r,r} = 0 \text{ for } i \neq r \ (i,r \in \{-1,0,1\}).$$

By condition (1.3.3) we have moreover

$$(\pi_{f,i})_{r,s} = 0 \quad \text{if } \ s-r = 1.$$

Indeed, for $(r,s) = (0,1)$ we have

$$\mathrm{Ext}^1(\mathbf{Q}_{l,S}(-1)[2], \textstyle\bigoplus_j(\iota_j)_*L_j[1]) = \textstyle\bigoplus_j H^0(C_j^o, L_j)(1) = 0.$$

For $(r,s) = (-1,0)$, we can use duality since $L_j(1)$ is self-dual. So we get for $i = -1,0,1$

$$\pi_{f,i} = (\pi_{f,i})_{i,i} + (\pi_{f,i})_{-1,1}.$$

It is then easy to see that the condition $\pi_{f,0}\circ\pi_{f,0} = \pi_{f,0}$ implies

$$(\pi_{f,0})_{-1,1} = 0, \text{ i.e. } \pi_{f,0} = (\pi_{f,0})_{0,0}.$$

In particular, $\pi_{f,0}$ is *unique*. Note that $(\pi_{f,1})_{-1,1} + (\pi_{f,-1})_{-1,1} = 0$ by $\pi_{f,-1}\circ\pi_{f,1} = 0$, and $(\pi_{f,i})_{-1,1}$ for $|i| = 1$ gives the ambiguity of the decomposition. Indeed, for any

$\eta \in \mathrm{Ext}^2(\mathbf{Q}_{l,S}(-1)[2], \mathbf{Q}_{l,S}[2])$, we can replace $\pi_{f,1}$, $\pi_{f,-1}$ with $\pi_{f,1}+\eta$ and $\pi_{f,-1}-\eta$ respectively. (If we assume the self-duality of the decomposition, this imposes some condition on the ambiguity.)

1.8. Remark. In case the base field is $\mathbf{C}$, the above argument can be justified. Indeed, let $d_X = \dim X$ and $Y = X \times_S X$ with the projections $pr_i : Y \to X$. Let $\mathbf{D}_Y$ denote the dualizing complex. Then, using the adjunction and the base change in [20], we have the isomorphisms (see also [9])

$$\begin{aligned}
\mathrm{End}_{D^b\mathrm{MHM}(S)}(\mathbf{R}f_*\mathbf{Q}_X) &= \mathrm{Hom}_{D^b\mathrm{MHM}(X)}(\mathbf{Q}_X, f^!\mathbf{R}f_*\mathbf{Q}_X)\\
&= \mathrm{Hom}_{D^b\mathrm{MHM}(X)}(\mathbf{Q}_X, \mathbf{R}(pr_1)_*pr_2^!\mathbf{Q}_X)\\
&= \mathrm{Hom}_{D^b\mathrm{MHM}(Y)}(pr_1^*\mathbf{Q}_X, pr_2^!\mathbf{Q}_X)\\
&= \mathrm{Ext}^{-2d_X}_{D^b\mathrm{MHS}}(\mathbf{Q}, \mathbf{R}\Gamma(Y, \mathbf{D}_Y(-d_X))).
\end{aligned}$$

Here MHS and $\mathrm{MHM}(X)$ denote respectively the categories of polarizable mixed Hodge structures [10] and mixed Hodge modules on X [20]. We have moreover the following

1.9. Proposition. *Let Y be a complex algebraic variety such that* $\dim \mathrm{Sing}\, Y \le d_Y - 2$ *where $d_Y = \dim Y$. Then we have an isomorphism*

$$\mathrm{CH}^1(Y)_{\mathbf{Q}} = \mathrm{Ext}^{2-2d_Y}_{D^b\mathrm{MHS}}(\mathbf{Q}, \mathbf{R}\Gamma(Y, \mathbf{D}_Y(1-d_Y))).$$

Proof. Let $Z = \mathrm{Sing}\, Y$ and $U = Y \setminus Z$ with the inclusions $i : Z \to Y$ and $j : U \to Y$. Since $\dim Z \le \dim Y - 2$, we have

$$\mathrm{CH}^1(Y) = \mathrm{CH}^1(U).$$

On the other hand, there is a distinguished triangle in $D^b\mathrm{MHM}(Y)$

$$i_*\mathbf{D}_Z \to \mathbf{D}_Y \to \mathbf{R}j_*\mathbf{D}_U \to,$$

inducing a long exact sequence of extension groups $\mathrm{Ext}^i_{D^b\mathrm{MHS}}(\mathbf{Q}, \mathbf{R}\Gamma(Y, *))$, and

$$\mathrm{Ext}^{-i}_{D^b\mathrm{MHS}}(\mathbf{Q}, \mathbf{R}\Gamma(Z, \mathbf{D}_Z(1-d_Y))) = 0 \quad \text{for } i > 2\dim Z,$$

since

$$H_i^{\mathrm{BM}}(Z) = H^{-i}(Z, \mathbf{D}_Z) = 0 \quad \text{for } i > 2\dim Z.$$

So the assertion is reduced to the smooth case, and follows from [21], Prop. 3.4. This finishes the proof of Proposition (1.9).

2. Proof of main theorems

2.1. Lemma. *With the notation of* (1.5)*, assume* f, g *are flat. Set* $n = \dim X - \dim S$*. Let* $\xi \in \mathrm{Cor}_S^i(X,S) = \mathrm{CH}^{n+i}(X)_{\mathbf{Q}}$ *and* $\xi' \in \mathrm{Cor}_S^j(S,Y) = \mathrm{CH}^j(Y)_{\mathbf{Q}}$*. Let* $pr_1 : X \times_S Y \to X$ *and* $pr_2 : X \times_S Y \to Y$ *denote the projections. Then the composition* $\xi' \circ \xi \in \mathrm{Cor}_S^{i+j}(X,Y) = \mathrm{CH}^{n+i+j}(X \times_S Y)_{\mathbf{Q}}$ *is given by* $pr_1^*\xi$ *if* $\xi' = [Y]$*, and* $pr_2^*\xi'$ *if* $\xi = [X]$*.*

Proof. The flatness of f, g implies that $X \times_S Y \to X \times_k Y$ is a regular embedding and the pr_i are flat. Moreover, we have locally a regular sequence defining $X \times_S Y$ in $X \times_k Y$ and it is a regular sequence for the pull-back by $X \times_k Y \to Y$ of any $\mathcal{O}_Y$-module. The last assertion follows from the flatness of pr_2 together with the theory of regular sequences (see e.g. [23], p. 71) since the Koszul complex calculates the pull-back by the embedding $X \times_S Y \to X \times_k Y$. So the assertion follows.

2.2. Lemma. *With the notation of* (1.5)*, let* $\xi \in \mathrm{Cor}_S^i(S,X) = \mathrm{CH}^i(X)_{\mathbf{Q}}$ *and* $\xi' \in \mathrm{Cor}_S^j(X,S) = \mathrm{CH}^{j+n}(X)_{\mathbf{Q}}$ *where* $n = \dim X - \dim S$*. Then* $\xi' \circ \xi \in \mathrm{Cor}_S^{i+j}(S,S) = \mathrm{CH}^{i+j}(S)_{\mathbf{Q}}$ *is given by* $f_*(\xi \cdot \xi') \in \mathrm{CH}^{i+j}(S)_{\mathbf{Q}}$*, where* $\xi \cdot \xi'$ *is the intersection of cycles on* X*.*

Proof. This immediately follows from the definition of the composition in (1.5).

2.3. Lemma. *With the notation of* (1.1)*, let* $\xi, \xi' \in \mathrm{Cor}_S^0(X,X) = \mathrm{CH}^1(X \times_S X)_{\mathbf{Q}}$ *which are represented by cycles supported in the inverse images of curves* C *and* C' *respectively on* S*. Assume* $\dim C \cap C' \le \dim S - 2$ *or one of the cycles belongs to* $pr^*\mathrm{CH}^1(S)_{\mathbf{Q}}$ *where* $pr : X \times_S X \to S$ *is the projection. Then their composition vanishes.*

Proof. If the second assumption is satisfied, we may assume that $\dim C \cap C' \le \dim S - 2$ by the moving lemma on S, since one of the cycles comes from S. Then the composition in $\mathrm{CH}^1(X \times_S X)_{\mathbf{Q}}$ is represented by a cycle supported in the inverse image of $C \cap C'$ which has codimension 2. So it vanishes. This finishes the proof of Lemma (2.3).

2.4. Proof of Theorem 2. We first assume that k is algebraically closed. Take any $\xi \in \mathrm{CH}^1(X)_{\mathbf{Q}}$ such that $f_*\xi = [S]$, i.e. its restriction to the generic fiber of f is a zero-cycle of degree 1. The ambiguity of ξ is given by $f^*\eta$ for $\eta \in \mathrm{CH}^1(S)_{\mathbf{Q}}$ since $f^{-1}(D)$ is irreducible for any irreducible curve D on S by (1.1.1). If $s \notin \mathrm{Sing}\, C$, there is an open neighborhood U of s such that the restriction of ξ over U is represented by $[Z]/2$, where Z is finite étale of degree 2 over U since f is a conic bundle. Set

$$p = pr_1^*\xi \in \mathrm{Cor}_S^0(X,X) = \mathrm{CH}^1(X \times_S X)_{\mathbf{Q}} \quad \text{so that} \quad {}^t p = pr_2^*\xi,$$

where pr_i is the i-th projection. By Lemma (2.1), we have

$$p = [X] \circ \xi.$$

where $\xi \in \mathrm{Cor}_S^0(X,S) = \mathrm{CH}^1(X)_{\mathbf{Q}}$ and $[X] \in \mathrm{Cor}_S^0(S,X) = \mathrm{CH}^0(X)_{\mathbf{Q}}$. Then p and ${}^t p$ are idempotents since we have by Lemma (2.2)

$$\xi \circ [X] = id \in \mathrm{Cor}_S^0(S,S). \tag{2.4.1}$$

We have ${}^t p\circ p=0$ since ${}^t[X]\circ[X]=0$ in $\mathrm{Cor}_S^{-1}(S,S)=0$. Note that $p\circ{}^t p=pr^*\eta$ with $\eta=f_*(\xi\cdot\xi)\in\mathrm{CH}^1(S)_{\mathbf{Q}}$ by Lemmas (2.1) and (2.2), where $pr:X\times_S X\to S$ is the projection. So we can define

$$\pi_{f,-1}=p\circ(1-{}^t p/2),\quad \pi_{f,1}=(1-p/2)\circ{}^t p.$$

Indeed, setting $\pi_{f,-1}=p\circ(1-a\,{}^t p)$ and $\pi_{f,1}=(1-b\,p)\circ{}^t p$ with $a,b\in\mathbf{Q}$, we get $a+b=1$ from the condition $\pi_{f,-1}\circ\pi_{f,1}=0$, and $a=b=1/2$ from the self-duality. Note that $\pi_{f,-1}$ and $\pi_{f,1}$ are still of the form $pr_1^*\xi$ and $pr_2^*\xi$ respectively, replacing ξ with $\xi-f^*\eta/2$. So we get isomorphisms of relative Chow motives

$$(X,\pi_{f,-1})=(S,\Delta_S),\quad (X,\pi_{f,1})=(S,\Delta_S)(-1),\tag{2.4.2}$$

induced by $\xi\in\mathrm{Cor}_S^0(X,S)$ and ${}^t[X]\in\mathrm{Cor}_S^{-1}(X,S)$ with inverse $[X]\in\mathrm{Cor}_S^0(S,X)$ and ${}^t\xi\in\mathrm{Cor}_S^1(S,X)$ respectively.

Let $\widetilde{X}_j\subset\widetilde{X_C}$ be the inverse image of C_j and let $g_j:\widetilde{X}_j\to X$, $p_j:\widetilde{X}_j\to D_j$ be natural morphisms. Set

$$\gamma_j:=(p_j)_*\circ(g_j)^*\in\mathrm{Cor}_S^{-1}(X,D_j),\quad \gamma'_j:=-{}^t\gamma_j/2\in\mathrm{Cor}_S^1(D_j,X).$$

Let σ_j be the involution of D_j associated with the double covering D_j/C'_j. This is identified with a cycle defined by its graph. The projector $\pi_{f,0,j}$ corresponding to C'_j is defined as in [19] by

$$\pi_{f,0,j}=\gamma'_j\circ\widetilde{\pi}_j\circ\gamma_j\quad\text{with}\quad \widetilde{\pi}_j:=(id-\sigma_j)/2.$$

This is represented by a cycle supported in $pr^{-1}(C_j)$, but does not belong to $pr^*\mathrm{CH}^1(S)_{\mathbf{Q}}$. More precisely, $\widetilde{X}_j\times_S\widetilde{X}_j$ has two irreducible components corresponding to the compositions of correspondences

$$(p_j)^*\circ id\circ(p_j)_*\ \text{ and }\ (p_j)^*\circ\sigma_j\circ(p_j)_*.$$

Taking further the composition with $(g_j)^*$ and $(g_j)_*$, we get the pushforward of these cycles by g_j.

By Proposition (2.5) below, $\gamma_j\circ{}^t\gamma_j\in\mathrm{Cor}_S^0(D_j,D_j)=\mathrm{Cor}^0_{C_j^o}(D_j^o,D_j^o)$ is expressed by the matrix

$$A:=\begin{pmatrix}-1&1\\1&-1\end{pmatrix}.$$

Here $D_j^o\to C_j^o$ is the restriction of $\rho_j:D_j\to C'_j$ over C_j^o; it is étale of degree 2.

On the other hand, $\widetilde{\pi}_j:=(id-\sigma_j)/2$ is expressed by the matrix

$$-\frac{1}{2}A=\begin{pmatrix}1/2&-1/2\\-1/2&1/2\end{pmatrix}$$

and is an idempotent since $A^2=-2A$. Then $\pi_{f,0,j}$ is an idempotent using Proposition (2.5) below. We have moreover

$$\pi_{f,0,j}\circ\gamma'_j\circ\widetilde{\pi}_j\circ\gamma_j\circ\pi_{f,0,j}=\pi_{f,0,j},\quad \widetilde{\pi}_j\circ\gamma_j\circ\pi_{f,0,j}\circ\gamma'_j\circ\widetilde{\pi}_j=\widetilde{\pi}_j.$$

This induces an isomorphism of relative Chow motives over S

$$(X, \pi_{f,0,j}) = \mathrm{Prym}(D_j/C'_j)(-1). \tag{2.4.3}$$

Then, using the compatibility of (1.5.3) with the composition of correspondences, we get

$$(\pi_{f,0,j})_*({}^pR^0f_*(\mathbf{Q}_{l,X}[3])) = (\iota_j)_*L_j[1] \quad \text{in} \quad {}^pR^0f_*(\mathbf{Q}_{l,X}[3]),$$

i.e. the action of the idempotent $\pi_{f,0,j}$ on $(\iota_{j'})_*L_{j'}[1] \subset {}^pR^0f_*(\mathbf{Q}_{l,X}[3])$ is the identity if $j = j'$, and vanishes otherwise. The action of $\pi_{f,0,j}$ on ${}^pR^if_*(\mathbf{Q}_{l,X}[3])$ vanishes for $|i| = 1$, since $\pi_{f,0,j}$ is supported in the inverse image of C_j. Moreover it follows from Lemma (2.3) that

$$\pi_{f,0,j} \circ \pi_{f,0,j'} = 0 \quad \text{for } \ j \neq j'.$$

So we get the middle projector

$$\pi_{f,0} := \textstyle\bigoplus_j \pi_{f,0,j}.$$

Now we have to show

$$\pi_{f,0,j} \circ \pi_{f,i} = \pi_{f,i} \circ \pi_{f,0,j} = 0 \quad \text{for } \ |i| = 1, \tag{2.4.4}$$

$$\zeta := 1 - \textstyle\sum_{-1 \le i \le 1} \pi_{f,i} = 0 \quad \text{in } \ \mathrm{Cor}^0_S(X, X). \tag{2.4.5}$$

For (2.4.4) it is enough to show $\pi_{f,0,j} \circ \pi_{f,i} = 0$ by duality. Consider the composition of

$${}^t\xi \in \mathrm{Cor}^1_S(S, X) \ \text{ and } \ (id - \sigma_j) \circ \gamma_j \in \mathrm{Cor}^{-1}_S(X, D_j) \ \text{ in } \ \mathrm{Cor}^0_S(S, D_j) = \mathrm{CH}^0(D_j)_{\mathbf{Q}}.$$

Since D_j is irreducible by hypothesis, the composition is given by $a[D_j]$ with $a \in \mathbf{Q}$. So $\pi_{f,0,j} \circ \pi_{f,1}$ is the composition of

$${}^t[X] \in \mathrm{Cor}^{-1}_S(X, S) = \mathrm{CH}^0(X)_{\mathbf{Q}} \quad \text{and} \quad a[X_{C_j}] \in \mathrm{Cor}^1_S(S, X) = \mathrm{CH}^1(X)_{\mathbf{Q}},$$

and is equal to $pr^*(a[C_j]) \in \mathrm{CH}^1(X \times_S X)_{\mathbf{Q}}$, where $X_{C_j} = f^{-1}(C_j)$. This vanishes since its composition with $\pi_{f,0,j}$ does by Lemma (2.3).

The argument is easier for $\pi_{f,0,j} \circ \pi_{f,-1}$ since we get an element in $\mathrm{CH}^{-1}(D_j)_{\mathbf{Q}}$ by taking the composition of $[X] \in \mathrm{Cor}^0_S(S, X)$ and $(id - \sigma_j) \circ \gamma_j \in \mathrm{Cor}^{-1}_S(X, D_j)$.

For (2.4.5), it is enough to show that ζ is nilpotent since it is an idempotent. By restricting ζ over a generic point of S and using condition (1.1.1), ζ is of the form

$$\zeta = pr^*\eta + \textstyle\sum_j c_j \pi_{f,0,j} \quad \text{with } \ \eta \in \mathrm{CH}^1(S)_{\mathbf{Q}},\ c_j \in \mathbf{Q}.$$

(Indeed, $\mathrm{CH}^0(pr^{-1}(C_j))_{\mathbf{Q}}$ is 2-dimensional by (1.1.1), and is generated by $\pi_{f,0,j}$ modulo $pr^*\mathrm{CH}^0(C_j)_{\mathbf{Q}}$ since $\pi_{f,0,j} \notin pr^*\mathrm{CH}^0(C_j)_{\mathbf{Q}}$.) We get $c_j = 0$ since the action of ζ on $(\iota_j)_*L_j[1] \subset {}^pR^if_*(\mathbf{Q}_{l,X}[3])$ vanishes. Then (2.4.5) follows from Lemma (2.3).

To show the uniqueness of $\pi_{f,i}$, let $\widetilde{\pi}_{f,i}$ be other mutually orthogonal projectors whose action on the cohomological direct images is the same as $\pi_{f,i}$. Then $\widetilde{\pi}_{f,i} = \pi_{f,i}$ over a sufficiently small open subvariety of S. Hence we have by the same argument as above (using condition (1.1.1))

$$\widetilde{\pi}_{f,i} = \pi_{f,i} + pr^*\eta_i + \sum_j a_{i,j}\pi_{f,0,j} \quad \text{with } \eta_i \in \mathrm{CH}^1(S)_{\mathbf{Q}},\ a_{i,j} \in \mathbf{Q}.$$

We have $a_{i,j} = 0$ by looking at the action on ${}^pR^0 f_*(\mathbf{Q}_{l,X}[3])$. We also get $\eta_0 = 0$ by $\widetilde{\pi}_{f,0} \circ \widetilde{\pi}_{f,0} = \widetilde{\pi}_{f,0}$ together with Lemma (2.3). Moreover, $\eta_{-1} + \eta_1 = 0$ by $\widetilde{\pi}_{f,-1} \circ \widetilde{\pi}_{f,1} = 0$ since

$$pr^*\eta_{-1} \circ \pi_{f,1} = pr^*\eta_{-1}, \quad \pi_{f,-1} \circ pr^*\eta_1 = pr^*\eta_1.$$

(Indeed, for $\xi_1 \in \mathrm{Cor}^1_S(S,X) = \mathrm{CH}^1(X)_{\mathbf{Q}}$ and $\xi_2 \in \mathrm{Cor}^0_S(X,S) = \mathrm{CH}^1(X)_{\mathbf{Q}}$, we have $\xi_2 \circ \xi_1 = f_*(\xi_1 \cdot \xi_2) \in \mathrm{CH}^1(S)_{\mathbf{Q}}$ by Lemma (2.2), and this is η in case $\xi_1 = \xi$ and $\xi_2 = f^*\eta$ since we can take a good representative of ξ as remarked at the beginning of this subsection. So the above equalities follow from Lemma (2.1).) Then the self-duality implies $\eta_{-1} = \eta_1 = 0$, and the uniqueness of the decomposition follows. (As for the ambiguity of ξ in the construction of $\pi_{f,\pm 1}$, we have also the following: If we replace ξ with $\xi + f^*\zeta$, then $\eta = f_*(\xi \cdot \xi)$ is replaced by $\eta + 2\zeta$, and hence $\pi_{f,-1}$ and $\pi_{f,1}$ are unchanged.)

Thus Theorem 2 is proved in the case $k = \bar{k}$. The assertion in the case $k \neq \bar{k}$ is reduced to the case $k = \bar{k}$ since the construction of the relative Chow-Künneth projectors is compatible with the base change although the decomposition of the middle projector becomes finer after the base change. So Theorem 2 follows.

To complete the proof of Theorem 2 we have to show the following. (In case C is smooth and irreducible, this also follows from [9], Example. 5.18.)

2.5. Proposition. *With the above notation,* $\gamma_j \circ {}^t\gamma_j \in \mathrm{Cor}^0_S(D_j, D_j) = \mathrm{Cor}^0_{C^o_j}(D^o_j, D^o_j)$ *is expressed by the matrix* A.

Proof. Take a sufficiently general closed point s of C^o_j. For $s' \in D_j$ lying over s, let $\widetilde{X}_{s'}$ denote the irreducible component of X_s corresponding to s' (this is identified with $p_j^{-1}(s') \subset \widetilde{X}_j$). Let T be a sufficiently general transversal slice to C^o_j at s, which is defined by

(2.5.1) $$T = h^{-1}(c) \setminus (C_j \setminus \{s\}) \ \text{ for a sufficiently general } c \in k,$$

where $T \cap C_j = \{s\}$ and h is a function defined on a non-empty open subvariety U of S such that $dh \neq 0$ on U and $dh|_{U \cap T} \neq 0$ on $U \cap T$. Let $\overline{X_T}$ be a smooth compactification of $X_T := f^{-1}(T)$ (this exists since it is 2-dimensional). The intersection matrix of $\widetilde{X}_{s'}, \widetilde{X}_{s''}$ in $\overline{X_T}$ (where s', s'' are the points of D_j over $s \in C^o_j$) is given by the matrix A since $[X_s] \cdot [\widetilde{X}_{s'}] = 0$ in $\overline{X_T}$ where we may assume that $f_T : X_T \to T$ is extended to $\overline{X_T} \to \overline{T}$.

As we have the injection

$$\mathrm{Cor}^0_S(D_j\, D_j) \subset \mathrm{End}((\rho_j)_*\mathbf{Q}_l),$$

where $\rho_j : D_j \to C_j$ is the projection, it suffices to calculate the composition

$$(\rho_j)_*\mathbf{Q}_l \xrightarrow{{}^t\gamma_j} R^2 f_*\mathbf{Q}_l(1) \xrightarrow{\gamma_j} (\rho_j)_*\mathbf{Q}_l.$$

Here the first morphism naturally factors through

$$ {}^t\gamma_j : (\rho_j)_*\mathbf{Q}_l \to \mathcal{H}^2_C\mathbf{R}f_*\mathbf{Q}_l(1), $$

which is the dual of the last morphism, where $\mathcal{H}^2_C$ is the local cohomology sheaf.

Restricting these to the transversal slice T, we obtain

$$ (2.5.2) \qquad (\gamma_j)_T \circ ({}^t\gamma_j)_T : \mathbf{Q}_{l,s'} \oplus \mathbf{Q}_{l,s''} \to R^2(f_T)_*\mathbf{Q}_l(1) \to \mathbf{Q}_{l,s'} \oplus \mathbf{Q}_{l,s''}, $$

where $f_T : X_T \to T$ is the restriction of f over T and similarly for $(\gamma_j)_T$, $({}^t\gamma_j)_T$. Here $\mathbf{Q}_{l,s'} \oplus \mathbf{Q}_{l,s''}$ is identified with a sheaf supported on s. The first morphism of (2.5.2) naturally factors through

$$ ({}^t\gamma_j)_T : \mathbf{Q}_{l,s'} \oplus \mathbf{Q}_{l,s''} \to \mathbf{H}^2_{\{s\}}(\mathbf{R}(f_T)_*\mathbf{Q}_l(1)). $$

By the generic base change theorem ([11], 2.9 and 2.10) this is the dual of the last morphism of (2.5.2) if $c \in k$ in (2.5.1) is sufficiently general. We have to show that (2.5.2) is expressed by the intersection matrix A.

For $t,\ u \in \{s', s''\}$, the (t,u)-component of (2.5.2) is given by the composition of morphisms of ℓ-adic cohomology groups

$$ H^0(\{t\}) \xrightarrow{p_j^*} H^0(\widetilde{X}_t) \xrightarrow{(\lambda_t)_*} H^2_c(X_T)(1) \to H^2(X_T)(1) \xrightarrow{(\lambda_u)^*} H^2(\widetilde{X}_u)(1) \xrightarrow{p_{j*}} H^0(\{u\}), $$

where $\lambda_t : \widetilde{X}_t \to X_T$ is the restriction of g_j, and similarly for $\lambda_u : \widetilde{X}_u \to X_T$. This is shown by using the commutative diagram

$$ \begin{array}{ccc} \mathbf{H}^2_{\{s\}}(K) & \to & (\mathcal{H}^2K)_s \\ \downarrow & & \uparrow \\ \mathbf{H}^2_c(T,K) & \to & \mathbf{H}^2(T,K), \end{array} $$

where $K = \mathbf{R}(f_T)_*\mathbf{Q}_l(1)$ so that $\mathbf{H}^2_c(T,K) = H^2_c(X_T)(1)$, etc.

Moreover the middle morphism $H^2_c(X_T)(1) \to H^2(X_T)(1)$ naturally factors through $H^2(\overline{X_T})(1)$, and hence we can replace X_T with $\overline{X_T}$ in the above composition of morphisms. This implies that (2.5.2) is expressed by the intersection matrix A as is desired. So Proposition (2.5) follows.

2.6. Proof of Theorem 1. With the notation of (2.4), we have

$$ \pi_{f,-1} = [X] \circ \xi, \quad \pi_{f,1} = {}^t\xi \circ {}^t[X]. $$

Let $\pi_{S,i}$ be the Chow-Künneth decomposition for S in [17] where $\pi_{S,i} = 0$ for $i \notin [0,4]$. We may assume the self-duality $\pi_{S,i} = {}^t\pi_{S,4-i}$ as is well-known (by the same argument as in the construction of $\pi_{f,\pm 1}$ in (2.4)). Define

$$ \pi_{X,i} = [X] \circ \pi_{S,i} \circ \xi + {}^t\xi \circ \pi_{S,i-2} \circ {}^t[X] + \delta_{i,3}\,\pi_{f,0}, $$

where $\delta_{i,3} = 1$ if $i = 3$, and 0 otherwise. Then we have isomorphisms of Chow motives

$$(X, [X] \circ \pi_{S,i} \circ \xi) = (S, \pi_{S,i}), \quad (X, {}^t\xi \circ \pi_{S,i-2} \circ {}^t[X]) = (S, \pi_{S,i-2})(-1),$$

using $\xi \circ [X] = id$ as in (2.4.1–2). Put $M_{0,j} = (X, \pi_{f,0,j})$. Using condition (1.1.1) together with duality we obtain

$$H^i(M_{0,j}) \cong H^{i-2}(C_j, (\iota_j)_* L_j)(-1) = 0$$

for all $i \neq 3$ in case $\bar{k} = k$, hence the motive $(X, \pi_{X,3})$ only has cohomology in degree 3. So we get the Chow-Künneth decomposition for X as desired.

2.7. Proof of Corollary 1. Using the action of correspondences on the Chow groups together with (2.4.4), we get

$$\mathrm{CH}^2_{\mathrm{alg}}(X)_{\mathbf{Q}} = \bigoplus\nolimits_{-1 \le i \le 1} (\pi_{f,i})_* \mathrm{CH}^2_{\mathrm{alg}}(X)_{\mathbf{Q}},$$

and

$$(\pi_{f,-1})_* \mathrm{CH}^2_{\mathrm{alg}}(X)_{\mathbf{Q}} = \mathrm{CH}^2_{\mathrm{alg}}(S)_{\mathbf{Q}}, \quad (\pi_{f,1})_* \mathrm{CH}^2_{\mathrm{alg}}(X)_{\mathbf{Q}} = \mathrm{CH}^1_{\mathrm{alg}}(S)_{\mathbf{Q}},$$

since $\xi \circ [X] = id$ as in (2.4.1). We have moreover

$$(\pi_{f,0})_* \mathrm{CH}^2_{\mathrm{alg}}(X)_{\mathbf{Q}} = \bigoplus\nolimits_j (\widetilde{\pi}_j)_* \mathrm{CH}^1_{\mathrm{alg}}(D_j)_{\mathbf{Q}} = \bigoplus\nolimits_j \mathrm{CH}^1_{\mathrm{alg}}(D_j)_{\mathbf{Q}}^{\sigma_j = -1},$$

where the last term is the (-1)-eigenspace of $\mathrm{CH}^1_{\mathrm{alg}}(D_j)_{\mathbf{Q}}$ for the action of σ_j. So the assertion is reduced to

$$\mathrm{CH}^1_{\mathrm{alg}}(D_j)_{\mathbf{Q}} = J(D_j)(k)_{\mathbf{Q}},$$

where $J(D_j)(k)$ is the abelian group of the k-valued points of the Picard variety of D_j/k. But this is well-known in case D_j has a k-valued point, and the general case is reduced to this case using the action of the Galois group and the group structure of the Picard variety. This finishes the proof of Corollary 1.

2.8. Relation with Murre's conjectures. Let $T(S) \subset \mathrm{CH}^2_{\mathrm{alg}}(S)$ be the Albanese kernel, and put $h^i(S) = (S, \pi_{S,i})$. Recall [18] that the rational Chow groups of the motives $h^i(S)$ are given by the table

	$h^0(S)$	$h^1(S)$	$h^2(S)$	$h^3(S)$	$h^4(S)$
CH^0	$\mathbf{Q}$	0	0	0	0
CH^1	0	$\mathrm{Pic}^0_{S/k}(k)_{\mathbf{Q}}$	$\mathrm{NS}(S)_{\mathbf{Q}}$	0	0
CH^2	0	0	$T(S)_{\mathbf{Q}}$	$\mathrm{Alb}_{S/k}(k)_{\mathbf{Q}}$	$\mathbf{Q}$.

Put $M_0 = (X, \pi_{f,0})$, and set $h^i(X) = (X, \pi_{X,i})$. As $h(X) \cong h(S) \oplus h(S)(-1) \oplus M_0$ we obtain

$$\begin{aligned}
h^0(X) &\cong h^0(S), \\
h^1(X) &\cong h^1(S), \\
h^2(X) &\cong h^0(S)(-1) \oplus h^2(S), \\
h^3(X) &\cong h^1(S)(-1) \oplus h^3(S) \oplus M_0, \\
h^4(X) &\cong h^2(S)(-1) \oplus h^4(S), \\
h^5(X) &\cong h^3(S)(-1), \\
h^6(X) &\cong h^4(S)(-1).
\end{aligned}$$

Hence the rational Chow groups of the motives $h^i(X)$ are given by the table

	$h^0(X)$	$h^1(X)$	$h^2(X)$	$h^3(X)$	$h^4(X)$	$h^5(X)$	$h^6(X)$
CH^0	$\mathbf{Q}$	0	0	0	0	0	0
CH^1	0	$\mathrm{Pic}^0_{S/k}(k)_{\mathbf{Q}}$	$\mathbf{Q}\oplus\mathrm{NS}(S)_{\mathbf{Q}}$	0	0	0	0
CH^2	0	0	$T(S)_{\mathbf{Q}}$	$A_{\mathbf{Q}}$	$\mathrm{NS}(S)_{\mathbf{Q}}\oplus\mathbf{Q}$	0	0
CH^3	0	0	0	0	$T(S)_{\mathbf{Q}}$	$\mathrm{Alb}_{S/k}(k)_{\mathbf{Q}}$	$\mathbf{Q}$

with

$$A_{\mathbf{Q}} = \mathrm{Pic}^0_{S/k}(k)_{\mathbf{Q}} \oplus \mathrm{Alb}_{S/k}(k)_{\mathbf{Q}} \oplus \mathcal{P}_X(k)_{\mathbf{Q}}.$$

The above table shows that the only correspondences that act nontrivially on $\mathrm{CH}^j(X)_{\mathbf{Q}}$ are $\pi_{X,j},\ldots,\pi_{X,2j}$. Hence Murre's conjectures A and B [18] hold for the conic bundle X. This is a refinement of results of del Angel and Müller–Stach for uniruled threefolds [1]. At present, it is not clear whether X satisfies Murre's conjectures C and D.

2.9. Remark. The decomposition

$$h(X) \cong h(S) \oplus h(S)(-1) \oplus (\textstyle\bigoplus_j \mathrm{Prym}(D_j/C'_j)(-1))$$

implies that the motive $h(X)$ is finite dimensional (in the sense of Kimura–O'Sullivan) if $h(S)$ is finite dimensional.

References

[1] P. L. del Angel and S. Müller-Stach, Motives of uniruled 3-folds, Compos. Math. 112 (1998), 1–16.

[2] R. Barlow, Rational equivalence of zero cycles for some more surfaces with $p_g = 0$, Inv. Math. 79 (1985), 303–308.

[3] A. Beauville, Variétés de Prym et jacobiennes intermédiaires, Ann. Sci. Ecole Norm. Sup. (4) 10 (1977), 309–391.

[4] A. Beilinson, Height pairing between algebraic cycles, Lect. Notes in Math., vol. 1289, Springer, Berlin, 1987, pp. 1–26.

[5] A. Beilinson, J. Bernstein and P. Deligne, Faisceaux pervers, Astérisque, vol. 100, Soc. Math. France, Paris, 1982.

[6] M. Beltrametti, On the Chow group and the intermediate Jacobian of a conic bundle, Ann. Mat. Pura Appl. (4) 141 (1985), 331–351.

[7] S. Bloch, Lectures on algebraic cycles, Duke University Mathematical series 4, Durham, 1980.

[8] S. Bloch, A. Kas and D. Lieberman, Zero cycles on surfaces with $p_g = 0$, Compos. Math. 33 (1976), 135–145.

[9] A. Corti and M. Hanamura, Motivic decomposition and intersection Chow groups, I, Duke Math. J. 103 (2000), 459–522.

[10] P. Deligne, Théorie de Hodge II, Publ. Math. IHES, 40 (1971), 5–57.

[11] P. Deligne, Théorème de finitude en cohomologie l-adique, in SGA 4 1/2, Lect. Notes in Math., vol. 569, Springer, Berlin, 1977, 233–261.

[12] C. Deninger and J. P. Murre, Motivic decomposition of abelian schemes and the Fourier transform, J. Reine Angew. Math. 422 (1991), 201–219.
[13] W. Fulton, Intersection theory, Springer, Berlin, 1984.
[14] B. B. Gordon, M. Hanamura and J. P. Murre, Relative Chow-Künneth projectors for modular varieties, J. Reine Angew. Math. 558 (2003), 1–14.
[15] B. B. Gordon, M. Hanamura and J. P. Murre, Absolute Chow-Künneth projectors for modular varieties, J. Reine Angew. Math. 580 (2005), 139–155.
[16] D. Mumford, Rational equivalence of 0-cycles on surfaces, J. Math. Kyoto Univ. 9 (1969), 195–204.
[17] J. P. Murre, On the motive of an algebraic surface, J. Reine Angew. Math. 409 (1990), 190–204.
[18] J. P. Murre, On a conjectural filtration on Chow groups of an algebraic variety, Indag. Math. 4 (1993), 177–201.
[19] J. Nagel, On the motivic decomposition conjecture for conic bundles, preprint.
[20] M. Saito, Mixed Hodge modules, Publ. RIMS, Kyoto Univ. 26 (1990), 221–333.
[21] M. Saito, Hodge conjecture and mixed motives, I, Proc. Sympos. Pure Math., 53, Amer. Math. Soc., Providence, RI, 1991, pp. 283–303.
[22] M. Saito, Chow-Künneth decomposition for varieties with low cohomological level, preprint (math.AG/0604254).
[23] J.-P. Serre, Algèbre locale, multiplicités, Lect. Notes in Math. 11, Springer, Berlin, 1975.

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