The Abel–Jacobi map for complete intersections

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1 Introduction

In an attempt to generalize the classical Noether–Lefschetz theorem for surfaces of degree $d \geq 4$ in $\mathbb{P}^3$, Griffiths and Harris [G-H] raised a number of questions concerning the behaviour of curves on a very general threefold X of degree $d \geq 6$ in $\mathbb{P}^4$. One of their questions is whether the image of the Abel–Jacobi map $\psi_X : \mathrm{CH}^2_{\mathrm{hom}}(X) \to J^2(X)$ is zero. Green [G2] and Voisin [V1] partially solved this problem; they showed that the image of ψ_X is contained in the torsion points of $J^2(X)$. A similar statement holds for odd-dimensional hypersurfaces in projective space: if $X = V(d) \subset \mathbb{P}^{2m}$ $(m \geq 2)$ is a very general hypersurface of degree $d \geq 4 + 2/(m-1)$, then the image of the Abel–Jacobi map ψ_X is contained in the torsion points of $J^m(X)$; see [G2].

We extend the result of Green and Voisin to smooth complete intersections of odd dimension in projective space (Theorem 4.1). In all but one of the cases where the conditions of Theorem 4.1 are not satisfied, it is known that the image of the Abel–Jacobi map is indeed non–torsion for a very general member of the family of complete intersections under consideration. The remaining exceptional case will be dealt with later.

To extend the result of Green and Voisin, we have to find an efficient algebraic description of the variable cohomology of complete intersections, analogous to the Jacobi ring description in the case of projective hypersurfaces. This problem has been solved through the work of various people, including Terasoma [Te], Konno [Ko], Libgober–Teitelbaum [L], [L-T] and Dimca [Di2]. The starting point is the following observation, due to Terasoma: if $X = V(f_0, \ldots, f_r)$ is a smooth complete intersection of multidegree $(d_0, \ldots, d_r)$ in $\mathbb{P}^{n+r+1}$ with $d_0 = \ldots = d_r = d$, then the variable cohomology of X is isomorphic (up to a shift in the Hodge filtration) to the variable cohomology of the hypersurface $\mathcal{X} = V(F) \subset \mathbb{P}^r \times \mathbb{P}^{n+r+1}$ of type $(1, d)$ defined by the bihomogeneous polynomial

$$F(x, y) = y_0 f_0(x) + \ldots + y_r f_r(x).$$

Konno extended this approach to the case of arbitrary multidegree by viewing $(f_0, \ldots, f_r)$ as a section of the vector bundle $E = \mathcal{O}_{\mathbb{P}}(d_0) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}}(d_r)$. The product of projective spaces is replaced by the projective bundle $\mathbb{P}(E^\vee)$, and $\mathcal{X}$ is replaced by the zero locus of the associated section of the tautological line bundle ξ_E on $\mathbb{P}(E^\vee)$. The variable cohomology is then described via the pseudo–Jacobi ring introduced by Green [G1]. Working with an additional hypothesis, Libgober obtained a description of the variable cohomology via residues of differential forms defined on $\mathbb{P}^{n+r+1}$, in the spirit of the work of Griffiths [Gr]; he observes that the variable cohomology is related to a quotient of the ring $S = \mathbb{C}[x_0, \ldots, x_{n+r+1}, y_0, \ldots, y_r]$, where S carries a suitable bigrading.

These different approaches were elegantly combined in the recent work of Dimca. He observes that $\mathbb{P}(E^\vee)$, being a smooth and compact toric variety, can be constructed as a geometric quotient. This explains the bigrading on S and shows that $\mathbb{P}(E^\vee)$ behaves like the ordinary projective space in many ways. Given this, Terasoma's original approach goes through with only minor modifications.

This paper is oranized as follows: in Section 2, Dimca's method is used to give a description of the variable cohomology in terms of the Jacobi ring of $\mathcal{X}$ in $\mathbb{P}(E^\vee)$. Next we discuss the so–called symmetrizer lemma in Section 3, using which we prove our main result in Section 4.

2 Description of the Jacobi ring

Let $X = V(d_0, \ldots, d_r) \subset \mathbb{P}^{n+r+1}$ be a smooth complete intersection of dimension $n \geq 3$, where $d_i \geq 2$ for $i = 0, \ldots, r$. We assume for the moment that $r > 0$, i.e., X is not a hypersurface. The $(r+1)$–tuple $(f_0, \ldots, f_r)$ of equations that define X represents a global section of the vector bundle $E = \mathcal{O}_{\mathbb{P}}(d_0) \oplus \ldots \mathcal{O}_{\mathbb{P}}(d_r)$. Let $P = \mathbb{P}(E^\vee)$ be the projective bundle whose fiber over a point $x \in \mathbb{P}^{n+r+1}$ is the projective space of hyperplanes in E_x. Using the results in [Cox], we can associate to the smooth and compact toric variety P its 'homogeneous coordinate ring' $S = \mathbb{C}[x_0, \ldots, x_{n+r+1}, y_0, \ldots, y_r]$. This ring carries a natural grading by elements of $\mathrm{Pic}(P) = \mathrm{Pic}(\mathbb{P}^{n+r+1}) \times \mathbb{Z} \cong \mathbb{Z}^2$. The variables x_i $(i = 0, \ldots, n+r+1)$ have bidegree $(0,1)$; the variables y_j $(j = 0, \ldots, r)$ have bidegree $(1, -d_j)$. Set

$$F(x,y) = y_0 f_0(x) + \ldots + y_r f_r(x),$$

and let $\mathcal{X} \subset P$ be the divisor defined by $F(x,y) \in S_{1,0}$.

Remark 2.1. Set $N = n+r+1$. We denote the open subset $\mathbb{C}^N \setminus \{0\} \times \mathbb{C}^{r+1} \setminus \{0\} \subset \mathbb{C}^{N+r+1}$ by U. The presence of a bigrading on the ring S is a consequence of the construction of P as a geometric quotient U/G, where $G = \mathbb{C}^* \times \mathbb{C}^*$ acts on U by

$$(t_1, t_2).(x_0, \ldots, x_N, y_0, \ldots, y_r) = (t_2 x_0, \ldots, t_2 x_N, t_2^{-d_0} t_1 y_0, \ldots, t_2^{-d_r} t_1 y_r).$$

Good references for this construction are [Cox], [Bat] and [Bat-Cox].

Lemma 2.2 *$\mathcal{X}$ is a very ample divisor on P if and only if $d_i > 0$ for all $i = 0, \ldots, r$.*

Proof: Note that $F(x,y) \in S_{1,0}$ represents the global section of the tautological line bundle $\xi_E = \mathcal{O}_P(1)$ that corresponds to $(f_0, \ldots, f_r)$ under the canonical isomorphism $H^0(P, \xi_E) = H^0(\mathbb{P}^n, E)$. It is readily verified that ξ_E is very ample if and only if the line bundles $\mathcal{O}_{\mathbb{P}}(d_i)$ are very ample for all $i = 0, \ldots, r$, cf. [B-S, (3.2.3)]. $\square$

Remark 2.3 To prove the above lemma from a toric point of view, one chooses a suitable divisor D of bidegree $(1,0)$ and shows that the support function ψ_D of the corresponding line bundle $\mathcal{O}(D)$ is strictly upper convex if and only if all the degrees d_i are positive.

Lemma 2.4 *X is non–singular $\Longleftrightarrow$ $\mathcal{X}$ is non–singular.*

Proof: Since $\frac{\partial F}{\partial y_i} = f_i(x)$ and $\partial F_o ver \partial x_i = \sum_{j=0}^r y_j \frac{\partial f_j}{\partial x_i}$, a point $(x,y) \in Y$ is singular if and only if $f_0(x) = \ldots = f_r(x) = 0$ and the matrix $(\frac{\partial f_j}{\partial x_i}(x))_{i,j}$ jas rank at most r, i.e., if and only if $x \in X$ is singular. $\square$

Definition 2.5 *Let i (resp. j) be the inclusion of X in $\mathbb{P}^{n+r+1}$ (resp. the inclusion of $\mathcal{X}$ in P). We define the variable cohomology of X and $\mathcal{X}$ as*

$$\begin{aligned} H^n_{\text{var}}(X,\mathbb{Q}) &= \operatorname{coker}(H^n(\mathbb{P}^{n+r+1},\mathbb{Q})) \xrightarrow{i^*} H^n(X,\mathbb{Q})) \\ H^{n+2r}_{\text{var}}(\mathcal{X},\mathbb{Q}) &= \operatorname{coker}(H^{n+2r}(P,\mathbb{Q}) \xrightarrow{j^*} H^{n+2r}(\mathcal{X},\mathbb{Q})). \end{aligned}$$

Remark 2.6 The notions of variable and primitive cohomology are strongly related: if V is a msooth projective variety and $D \subset V$ is a smooth complete intersection of ample divisors in V of dimension d, one can show that $H^d_{\text{pr}}(D) = i^* H^d_{\text{pr}}(V) \oplus H^d_{\text{var}}(D)$.

Let $\pi : P \to \mathbb{P}^{n+r+1}$ be the projection, and write $\tilde{X} = \pi^{-1}(X) = X \times \mathbb{P}^r$. Let $\iota : \mathcal{X} \to P$ be the inclusion map, and put $\varphi = \pi \circ \iota : \mathcal{X} \to \mathbb{P}^{n+r+1}$.

Lemma 2.7 *The inclusion $\tilde{X} \hookrightarrow \mathcal{X}$ induces an isomorphism of Hodge structures*

$$H^{n+2r}_{\text{var}}(\mathcal{X},\mathbb{C}) \xrightarrow{\sim} H^n_{\text{var}}(X,\mathbb{C}) \otimes H^{2r}(\mathbb{P}^r,\mathbb{C}).$$

Proof: Consider the Leray spectral sequences

$$\begin{aligned} E_2^{p,q} &= H^p(\mathbb{P}^{n+r+1}, R^q\pi_*\mathbb{C}) \Rightarrow H^{p+q}(P,\mathbb{C}) \\ {}'E_2^{p,q} &= H^p(\mathbb{P}^{n+r+1}, R^q\varphi_*\mathbb{C}) \Rightarrow H^{p+q}(\mathcal{X},\mathbb{C}) \end{aligned}$$

associated to the maps $\pi : P \to \mathbb{P}^{n+r+1}$ and $\varphi : \mathcal{X} \to \mathbb{P}^{n+r+1}$. Consider also the Künneth spectral sequence

$$''E_2^{p,q} = H^p(X,\mathbb{C}) \otimes H^q(\mathbb{P}^r,\mathbb{C}) \Rightarrow H^{p+q}(\tilde{X},\mathbb{C}).$$

The local systems $R^q\pi_*\mathbb{C}$ and $R^q\varphi_*\mathbb{C}$ can be described as follows:

$$\begin{aligned} R^q\pi_*\mathbb{C} &= \begin{cases} \mathbb{C}(-\frac{q}{2}) & \text{if } q \le 2r,\ q \text{ even} \\ 0 & \text{otherwise} \end{cases} \\ R^q\varphi_*\mathbb{C} &= \begin{cases} \mathbb{C}(-\frac{q}{2}) & \text{if } q < 2r,\ q \text{ even} \\ \mathbb{C}_X(-r) & \text{if } q = 2r \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

All the spectral sequences degenerate at E_2. The Lefschetz hyperplane theorem shows that

$$E_2^{p,q} \xrightarrow{\sim} {}'E_2^{p,q} \xrightarrow{\sim} {}''E_2^{p,q}$$

for $(p,q)\neq(n,2r)$ and

$$E_2^{n,2r}\hookrightarrow {}'E_2^{n,2r}\xrightarrow{\sim}{}''E_2^{n,2r}.$$

Hence

$$\begin{aligned}H^{n+2r}_{\mathrm{var}}(\mathcal{X},\mathbb{C}) &\cong \operatorname{coker}(E_2^{n,2r}\to {}'E_2^{n,2r})\\ &\cong \operatorname{coker}(E_2^{n,2r}\hookrightarrow {}''E_2^{n,2r})\\ &\cong H^n_{\mathrm{var}}(X,\mathbb{C})\otimes H^{2r}(\mathbb{P}^r,\mathbb{C}).\end{aligned}$$

For details, see [Ko] or [Te]. □

The description of the variable cohomology of $\mathcal{X}$ strongly resembles the description of the primitive cohomology of a hypersurface in $\mathbb{P}^{n+1}$. As most of the results are similar to those in [Di1], [Do] and [CGGH], we shall omit their proofs.

The Euler vector fields

$$e_1=\sum_{i=0}^{r} y_i\frac{\partial}{\partial y_i}$$

and

$$e_2=\sum_{i=0}^{n+r+1} x_i\frac{\partial}{\partial x_i}-\sum_{i=0}^{r} d_iy_i\frac{\partial}{\partial y_i}$$

generate the action of $G=\mathbb{C}^*\times\mathbb{C}^*$ on $U=\mathbb{C}^{n+r+1}\setminus\{0\}\times\mathbb{C}^{r+1}\setminus\{0\}$. The orbits of this action are the fibers of $\pi:U\to P$.

Definition 2.8.

(i) *For a monomial* $f=x_0^{\alpha_0}\dots x_N^{\alpha_N}y_0^{\beta_0}\dots y_r^{\beta_r}$ *we define* $|f|_1=\sum_{i=0}^r\beta_i$, $|f|_{2,x}=\sum_{i=0}^{n+r+1}\alpha_i$, $|f|_{2,y}=-\sum_{j=0}^r d_j\beta_j$ *and* $|f|_2=|f|_{2,x}+|f|_{2,y}$.

(ii) *For a differential form* $\omega=f.dx_{s_1}\wedge\dots\wedge dx_{s_i}\wedge dy_{t_1}\wedge\dots\wedge dy_{t_j}$ *we set* $|\omega|_1=|f|_1+j$, $|\omega|_{2,x}=|f|_{2,x}+i$, $|\omega|_{2,y}=|f|_{2,y}-\sum_{k=1}^j d_{t_k}$ *and* $|\omega|_2=|\omega|_{2,x}+|\omega|_{2,y}$.

In the statement of the following Lemma, the differential d is written in the form $d=d_x+d_y$. We write $A^k=H^0(\mathbb{C}^{n+2r+2},\Omega^k_{\mathbb{C}^{n+2r+2}})$, and denote the contraction with a vector field e by i_e.

Lemma 2.9. *If $\omega \in A^i$ and $\omega' \in A^j$, then*

(i) $i_e(\omega \wedge \omega') = i_e(\omega) \wedge \omega' + (-1)^i \omega \wedge i_e(\omega')$

(ii) $i_{e_1}(df) = |f|_1 f$, $i_{e_2}(df) = |f|_2 f$

(iii) $d_y(i_{e_1}\omega) + i_{e_1}(d_y\omega) = |\omega|_1\, \omega$

(iv) $d_x(i_{e_2}\omega) + i_{e_2}(d_x\omega) = |\omega|_{2,x}\, \omega$

(v) $d_y(i_{e_2}\omega) + i_{e_2}(d_y\omega) = |\omega|_{2,y}\, \omega$

(vi) $|i_{e_1}(\omega)|_k = |i_{e_2}(\omega)|_k = |\omega|_k$, $k = 1, 2$.

Lemma 2.10. *A rational k–form φ on U given by*

$$\varphi = \frac{1}{H(x,y)} \sum_{\substack{I,J \\ |I|+|J|=k}} A_{I,J}(x,y)\, dx_I \wedge dy_J$$

satisfies $\varphi = \pi^*\omega$ for a rational k–form ω on P if and only if

(i) *φ is G–invariant, i.e., $|\varphi|_1 = |\varphi|_2 = 0$.*

(ii) $i_{e_1}(\varphi) = i_{e_2}(\varphi) = 0$.

Proof: One easily checks that $\varphi = \pi^*\omega$ for a rational k–form ω on P if and only if φ and $d\varphi$ are horizontal, i.e., $i_e(\varphi) = i_e(d\varphi) = 0$ for all vertical vector fields e. This is equivalent to $i_{e_1}(\varphi) = i_{e_2}(\varphi) = i_{e_1}(d\varphi) = i_{e_2}(d\varphi) = 0$, hence the assertion follows from the previous Lemma. $\square$

From now on we shall identify rational differential forms on P with their pullbacks to U.

Lemma 2.11. *Suppose that $\psi \in A^k$ satisfies the following conditions*

(i) $i_{e_1}(\psi) = i_{e_2}(\psi) = 0$

(ii) *$|\psi|_1 \neq 0$ and at least one of $|\psi|_2$, $|\psi|_{2,x}$ is nonzero.*

Then $\psi = i_{e_2} i_{e_1}(\varphi)$ for some $\varphi \in A^{k+2}$.

Proof: If $|\psi|_1 = \alpha \neq 0$ and $|\psi|_{2,x} = \beta \neq 0$, we can write

$$\begin{aligned}
\alpha\beta\psi &= \alpha(i_{e_2}(d_x\psi) + d_x(i_{e_2}\psi)) \\
&= \alpha i_{e_2} d_x \psi = i_{e_2}(d_x(\alpha\psi)) \\
&= i_{e_2}(d_x(i_{e_1}(d_y\psi) + d_y(i_{e_1}\psi))) \\
&= i_{e_2}(d_x(i_{e_1}(d_y\psi))) = -i_{e_2} i_{e_1}(d_x d_y \psi).
\end{aligned}$$

$\square$

Corollary 2.12.

$$\psi \in H^0(P, \Omega_P^k(q\mathcal{X})) \iff \psi = \frac{i_{e_2} i_{e_1}(\varphi)}{F(x,y)^q}$$

for some $\varphi \in A^{k+2}$ *with* $|\varphi|_1 = q$ *and* $|\varphi|_2 = 0$.

The following Lemma shows how to express $d\psi$ in a similar form:

Lemma 2.13. *If*

$$\psi = \frac{i_{e_2} i_{e_1}(\varphi)}{F^q},$$

then

$$d\psi = \frac{i_{e_2} i_{e_1}(F d\varphi - q\, dF \wedge \varphi)}{F^{q+1}}.$$

Lemma 2.14.

(i)

$$\psi \in H^0(P, \Omega_P^{n+2r+1}((q+1)\mathcal{X})) \iff \psi = \frac{P(x,y)\Omega}{F^{q+1}}$$

where $\Omega = i_{e_2} i_{e_1}(dx_0 \wedge \ldots \wedge dx_{n+r+1} \wedge dy_0 \wedge \ldots \wedge dy_r)$.

(ii)

$$\tilde{\psi} \in H^0(P, \Omega_P^{n+2r}(q\mathcal{X}) \iff \tilde{\psi} = \frac{i_{e_2} i_{e_1}(\varphi)}{F^q}$$

where

$$\varphi = \sum_{i=0}^{n+r+1} Q_i(x,y)\Omega_i \wedge dy_0 \wedge \ldots \wedge dy_r + \sum_{\alpha=0}^{r} R_\alpha(x,y) dx_0 \wedge \ldots \wedge dx_n \wedge \Omega_\alpha,$$

$$\Omega_i = dx_0 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_{n+r+1},$$
$$\Omega_\alpha = dy_0 \wedge \ldots \wedge \widehat{dy_\alpha} \wedge \ldots \wedge dy_r.$$

Definition 2.15. *The Jacobi ideal $J(F) \subset S$ is the ideal in S generated by the partial derivatives*

$$\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_{n+r+1}}, \frac{\partial F}{\partial y_0}, \ldots, \frac{\partial F}{\partial y_r}.$$

The Jacobi ring R is the quotient ring $S/J(F)$. The bigrading on S induces a bigrading on R.

Proposition 2.16. *There is a natural isomorphism*

$$H^{n-p,p}_{\mathrm{var}}(X) \cong R_{p,d(X)},$$

where $n = \dim X$ and $d(X) = \sum_{i=0}^{r} d_i - n - r - 2$.

Proof: We have already seen that $H^{n-p,p}_{\mathrm{var}}(X) \cong H^{n-p+r,p+r}_{\mathrm{var}}(\mathcal{X})$. There is an exact sequence

$$0 \to H^{n-p+r+1,p+r}_{\mathrm{pr}}(P) \to H^{p+r}(\Omega_P^{n-p+r+1}(\log \mathcal{X})) \to H^{n-p+r,p+r}_{\mathrm{var}}(\mathcal{X}) \to 0.$$

The Leray–Hirsch theorem shows that

$$H^{n+2r}(P) \cong \bigoplus_{i+j=n+2r} H^i(\mathbb{P}^{n+r+1}) \otimes H^j(\mathbb{P}^{2r}.$$

Write $b_i(P) = \dim H^i(P, \mathbb{C})$. The above formula implies that $b_{n+2r-2}(P) = b_{n+2r}(P)$, so $H^{n+2r}_{\mathrm{pr}}(P) = 0$. Hence

$$H^{n-p+r,p+r}_{\mathrm{var}}(\mathcal{X}) \cong H^{p+r}(\Omega_P^{n-p+r+1}(\log \mathcal{X})).$$

Since $\mathcal{X} \subset P$ is an ample divisor by Lemma 2.2, we can apply the Bott vanishing theorem on P (see [Bat-Cox, Theorem 7.1]) to obtain

$$H^i(P, \Omega_P^j(k\mathcal{X})) = 0 \text{ for all } i > 0, k > 0 \text{ and } j \geq 0.$$

A spectral sequence argument shows that

$$H^{p+r}(\Omega_P^{n-p+r+1}(\log \mathcal{X})) \cong \frac{H^0(\Omega_P^d((p+r+1)\mathcal{X}))}{H^0(\Omega_P^d((p+r)\mathcal{X})) + dH^0(\Omega_P^{d-1}((p+r)\mathcal{X})),}$$

where $d = \dim P = n + 2r + 1$.

By Lemma 2.14, an element of $H^0(P, \Omega_P^d((p+r+1)\mathcal{X}))$ can be written in the form

$$\psi_P = \frac{P(x,y)\Omega}{F^{p+r+1}}$$

where $\deg \Omega = (r+1, -d(X))$ and $\deg(P(x,y)) = (p, d(X))$. What we have shown so far is that the map

$$\begin{array}{rcl} \mathrm{Res} : S_{p,d(X)} & \rightarrow & F^{d-p+r} H_{\mathrm{var}}^{n+2r+1}(\mathcal{X}) \\ P(x,y) & \mapsto & [\mathrm{Res}(\psi_P)] \end{array}$$

is surjective. If $\tilde{\psi} \in H^0(P, \Omega_P^{n+2r}((p+r)\mathcal{X}))$, then $\tilde{\psi} = \frac{i_{e_2} i_{e_1}(\varphi)}{F^{p+r}}$ and

$$d\tilde{\psi} = \frac{\{F(\sum_{i=0}^N \frac{\partial Q_i}{\partial x_i} + \sum_{j=0}^r \frac{\partial R_j}{\partial y_j}) - (p+r)(\sum_{i=0}^N \frac{\partial F}{\partial x_i} Q_i + \sum_{j=0}^r \frac{\partial F}{\partial y_j} R_j\}\Omega}{F^{p+r+1}},$$

where $N = n + r + 1$. Hence

$$\psi_P \equiv d\tilde{\psi} \bmod H^0(\Omega_P^{n+2r+1}((p+r)\mathcal{X}) \iff P \in J(F).$$

This shows that Res induces an isomorphism $R_{p,d(X)} \cong H_{\mathrm{var}}^{n-p+r,p+r}(\mathcal{X})$, as desired. $\square$

Remark 2.17.

(i) If $r = 0$, then $\mathbb{P}(E^\vee) \cong \mathbb{P}^{n+1}$, $S = \mathbb{C}[x_0, \ldots, x_{n+1}, y_0]$ and $F(x,y) = y_0 f_0(x)$. Clearly the ring $R(F)$ is different from the Jacobi ring $R(f_0)$ of the hypersurface $V(f_0) \subset \mathbb{P}^{n+1}$, but the map $\alpha : S \to \mathbb{C}[x_0, \ldots, x_{n+1}]$ that sends $G(x_0, \ldots, x_{n+1}, y_0)$ to $G(x_0, \ldots, x_{n+1}, 1)$ induces an isomorphism $R(F)_{p,d(X)} \xrightarrow{\sim} R(f_0)_{(p+1)d_0 - n-1}$ between the graded pieces of these rings that describe $H_{\mathrm{var}}^{n-p,p}(X)$.

(ii) The toric description of P shows that the bidegree of the canonical bundle K_P is $(-r-1, d(X))$ and $S_{p,d(X)} \cong H^0(P, K_P \otimes \xi_E^{p+r+1})$.

(iii) The description of the variable cohomology $H^n_{\mathrm{var}}(X)$ for a complete intersection X in an arbitrary smooth and compact toric variety $\mathbb{P}_\Sigma$ proceeds along the same lines. The number of Euler vector fields equals the rank of $\mathrm{Pic}(\mathbb{P}(E^\vee)) = \mathrm{Pic}(\mathbb{P}_\Sigma) \times \mathbb{Z}$.

3 Symmetrizer lemma

Using a version of the symmetrizer lemma, we prove that the infinitesimal invariants associated to certain normal functions are zero. Consequently these normal functions are torsion sections of the fiber space of intermediate Jacobians.

We keep the notation of Section 2, but from now on we consider the case where X is a smooth complete intersection in $\mathbb{P}^{2m+r}$ of odd dimension $n = 2m-1$. In this case we have $H^{2m-1}(X) = H^{2m-1}_{\mathrm{var}}(X) = H^{2m-1}_{\mathrm{pr}}(X)$. Let $U \subset \mathbb{P}H^0(\mathbb{P}^{2m+r}, E)$ be the open subset parametrizing smooth complete intersections, and let $f : X_U \to U$ be the universal family. The cohomology groups of the fibers of f give rise to a local system $H_\mathbb{Z} = R^{2m-1}f_*\mathbb{Z}$. Let $\mathcal{H}^{2m-1} = H_\mathbb{Z} \otimes_\mathbb{Z} \mathcal{O}_U$ be the associated Hodge bundle; it is filtered by holomorphic subbundles $\mathcal{F}^p$ $(0 \le p \le 2m-1)$. The Hodge bundle comes equipped with a flat connection ∇, the Gauss–Manin connection, whose flat sections are the sections of the local system $H_\mathbb{Z}$. The filtration of subbundles $\mathcal{F}^\bullet$ is shifted by ∇ according to the Griffiths transversality rule $\nabla \mathcal{F}^p \subset \Omega^1_U \otimes \mathcal{F}^{p-1}$. Let

$$\mathcal{J}^m = \mathcal{H}^{2m-1}/(\mathcal{F}^m + H_\mathbb{Z})$$

be the sheaf of intermediate Jacobians over U. The Gauss–Manin connection induces a map

$$\overline{\nabla} : \mathcal{J}^m \to \Omega^1_U \otimes \mathcal{H}^{2m-1}/\mathcal{F}^{m-1},$$

whose kernel is denoted by $\mathcal{J}^m_h$. By abuse of language, a global section of $\mathcal{J}^m_h$ is called a normal function.

To study the image of the Abel–Jacobi map using normal functions, we 'spread out' cycles on a very general fiber to relative cycles.

If $X_0 = V(d_0, \ldots, d_r) \subset Y$ is a very general complete intersection and $Z_0 \in Z^m_{\text{hom}}(X_0)$, there exist a finite étale covering $g : T \to U$, a relative cycle $Z_T \in CH^m_{\text{hom}}(X_T/T)$ and a point $t_0 \in g^{-1}(0)$ such that the fiber of Z_T over t_0 is Z_0; cf. [H]. Set $Z_t = Z_T \cap f_T^{-1}(t)$ and let $\nu \in H^0(T, \mathcal{J}^m_h)$ be the normal function given by $\nu(t) = \psi_{X_t}(Z_t)$. Set $H_{\mathbb{Q}} = H_{\mathbb{Z}} \otimes_Z \mathbb{Q}$. The twisted De Rham complex

$$\Omega^\bullet_T \otimes H_{\mathbb{Q}} : \mathcal{H} \to \Omega^1_T \otimes \mathcal{H} \to \Omega^2_T \otimes \mathcal{H} \to \cdots$$

is a resolution of the local system $H_{\mathbb{C}} = H_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$. Let $F^m(\Omega^\bullet_T \otimes H_{\mathbb{Q}})$ be the subcomplex

$$\mathcal{F}^m \to \Omega^1_T \otimes \mathcal{F}^{m-1} \to \Omega^2_T \otimes \mathcal{F}^{m-2} \to \cdots$$

of $\Omega^\bullet_T \otimes H_{\mathbb{Q}}$. Note that although the differential ∇ of $F^m(\Omega^\bullet_T \otimes H_{\mathbb{Q}})$ is not $\mathcal{O}_T$–linear, the induced differential $\overline{\nabla}$ on the graded pieces $\mathrm{Gr}^p_F(\Omega^\bullet_T \otimes H_{\mathbb{Q}})$ is $\mathcal{O}_T$–linear. The normal function ν has an infinitesimal invariant $\delta\nu \in H^0(T, \mathcal{H}^1(F^m(\Omega^\bullet_T \otimes H_{\mathbb{Q}})))$; see [G3, Lecture 6]. It is known that ν has flat local liftings if and only if $\delta_\nu = 0$ [loc. cit.]. Hence, to prove that $\delta\nu = 0$ it suffices to show that

$$\mathcal{H}^1(\mathrm{Gr}^p_F(\Omega^\bullet_T \otimes H_{\mathbb{Q}})) = 0 \text{ for all } p \geq m.$$

Let T_0 be the tangent space to U at $0 \in U$. As $g : T \to U$ is an étale covering, the tangent space to T at $t_0 \in g^{-1}(0)$ is isomorphic to T_0. We want to show that the first cohomology group of the complex

$$H^{p,2m-p-1}(X_0) \to T_0^\vee \otimes H^{p-1,2m-p}(X_0) \to \bigwedge^2 T_0^\vee \otimes H^{p-2,2m-p+1}(X_0)$$

vanishes. Dualizing this complex we obtain

$$\bigwedge^2 T_0 \otimes H^{2m-p+1,p-2}(X_0) \to T_0 \otimes H^{2m-p,p-1}(X_0) \to H^{2m-p-1,p}(X_0).$$

Lemma 3.1. *The diagram*

$$\begin{array}{ccccc}
\bigwedge^2 S_{1,0} \otimes R_{p-2,d(X)} & \to & S_{1,0} \otimes R_{p-1,d(X)} & \to & R_{p,d(X)} \\
\downarrow & & \downarrow & & \downarrow \\
\bigwedge^2 T_0 \otimes H^{2m-p+1,p-2}(X_0) & \to & T_0 \otimes H^{2m-p,p-1}(X_0) & \to & H^{2m-p-1,p}(X_0)
\end{array}$$

is commutative.

Proof: This is a standard consequence of the description of $H^{2m-1}(X)$ by residues of differential forms. Note that the identification of the tangent space $T_{U,0}$ with $S_{1,0}$ is obtained by sending a polynomial $G(x,y) \in S_{1,0}$ to the infinitesimal deformation of $\mathcal{X}$ given by $F_t(x,y) = F(x,y) + t\,G(x,y)$, $t^2 = 0$. The commutativity of the right square is established by the following basic observation: if we write

$$\Omega_{P(t)} = \frac{P(t)\Omega}{(F+tG)^{p+r}},$$

then

$$\partial/\partial t\, \Omega_{P(t)}|_{t=0} \equiv -(p+r)\frac{P(0)G\Omega}{F^{p+r+1}}$$

modulo differential forms with poles of lower order. It follows that

$$\begin{aligned} \nabla_{\partial/\partial t} \operatorname{Res} \Omega_{P(t)} &= \partial/\partial t(\operatorname{Res} \Omega_{P(t)})|_{t=0} \\ &= \operatorname{Res}(\partial/\partial t\, \Omega_{P(t)}|_{t=0}) = \operatorname{Res}(\Omega_{PG}). \end{aligned}$$

The commutativity of the square on the left hand side follows in a similar way. $\square$

In the sequel we shall use some standard multi–index notation. For a multi–index $I = (i_0, \ldots, i_r)$ we write $\langle d, I\rangle = d_0 i_0 + \ldots + d_r i_r$. Let (i_0) denote the $(r+1)$–tuple $(0, \ldots, 0, 1, 0, \ldots, 0)$ where the number 1 occurs at position i_0.

Lemma 3.2. *The multiplication map*

$$S_{a,b} \otimes S_{\alpha,\beta} \longrightarrow S_{a+\alpha,b+\beta}$$

is surjective if

(i) $a \geq 0$, $\alpha \geq 0$

(ii) $\langle d, I\rangle + b \geq 0$ *for all* I *with* $|I| = a$, $\langle d, J\rangle + \beta \geq 0$ *for all* J *with* $|J| = \alpha$.

Proof: Note that $S_{a,b}$ is spanned by the monomials $y^I x^J$ with $|I| = a$, $|J| = \langle d, I\rangle + b$. Given a monomial $x^K y^L$ with $|L| = a+\alpha$, $|K| = \langle d, L\rangle + b + \beta$ we can write $L = L_1 \cup L_2$ with $|L_1| = a$, $|L_2| = \alpha$ and $K = K_1 \cup K_2$ where $|K_1| = \langle d, L_1\rangle + b$, $|K_2| = \langle d, L_2\rangle + \beta$. $\square$

Lemma 3.3. (symmetrizer lemma) *Assume that* $n \geq 2$*,* $p \geq 2$ *and* $d_0 \geq \ldots \geq d_r$*. The complex*

$$\bigwedge^2 S_{1,0} \otimes R_{p-2,d(X)} \to S_{1,0} \otimes R_{p-1,d(X)} \to R_{p,d(X)}$$

is exact at the middle term if the following two conditions are satisfied:

(*) $d_0 + \ldots + d_r + (p-2)d_r \geq n + r + 3$

(**) $d_1 + \ldots + d_r + (p-1)d_r \geq n + r + 2$.

To prove the symmetrizer lemma, it suffices to show that

(i) The complex

$$\bigwedge^2 S_{1,0} \otimes S_{p-2,d(X)} \xrightarrow{g} S_{1,0} \otimes S_{p-1,d(X)} \xrightarrow{h} S_{p,d(X)}$$

is exact at the middle term.

(ii) The map

$$S_{1,0} \otimes J_{p-1,d(X)} \to J_{p,d(X)}$$

is surjective.

This follows by chasing the commutative diagram with exact columns

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\bigwedge^2 S_{1,0} \otimes J_{p-2,d(X)} & \to & S_{1,0} \otimes J_{p-1,d(X)} & \to & J_{p,d(X)} \\
\downarrow & & \downarrow & & \downarrow \\
\bigwedge^2 S_{1,0} \otimes S_{p-2,d(X)} & \to & S_{1,0} \otimes S_{p-1,d(X)} & \to & S_{p,d(X)} \\
\downarrow & & \downarrow & & \downarrow \\
\bigwedge^2 S_{1,0} \otimes R_{p-2,d(X)} & \to & S_{1,0} \otimes R_{p-1,d(X)} & \to & R_{p,d(X)} \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0.
\end{array}$$

We shall verify the conditions (i) and (ii) in Lemmas 3.4 and 3.7.

Lemma 3.4. *The complex*

$$\bigwedge^2 S_{1,0} \otimes S_{p-2,k} \xrightarrow{g} S_{1,0} \otimes S_{p-1,k} \xrightarrow{h} S_{p,k}$$

is exact at the middle term provided that $p \geq 2$ and $\langle d, J\rangle + k > 0$ for all multi–indices J with $|J| = p - 2$.

Proof: The map g is given by

$$g(y_{i_0}x^{I_0} \wedge y_{i_1}x^{I_1} \otimes y^K x^L) = y_{i_0}x^{I_0} \otimes y^{K+(i_1)}x^{L+I_1} - y_{i_1}x^{I_1} \otimes y^{K+(i_0)}x^{L+I_0}.$$

This shows that

$$y_{i_0}x^{I_0} \otimes y^{J_0}x^{K_0} \equiv y_{i_1}x^{I_1} \otimes y^{J_1}x^{K_1} \mod (\operatorname{im} g)$$

if $J_0 + (i_0) = J_1 + (i_1)$, $K_0 + I_0 = K_1 + I_1$ and $K_1 - I_0 \geq 0$. In fact, if these conditions are satisfied it follows that $K_0 - I_1 \geq 0$ and

$$\begin{aligned} M = K_0 + I_0 = K_1 + I_1 &= I_0 + I_1 + (K_0 - I_1) \\ &= I_0 + I_1 + (K_1 - I_0); \end{aligned}$$

hence $K_0 - I_1 = K_1 - I_0 = L$, $|L| = \langle d, J\rangle + k$, and $J = J_0 - (i_1) = J_1 - (i_0)$. Combining the two relations

$$\begin{aligned} y_{i_0}x^{I_0} \otimes y^{J_0}x^{M-I_0} &\equiv y_{i_1}x^{M-L} \otimes y^{J_1}x^{L} \\ &\equiv y_{i_2}x^{I_2} \otimes y^{J_2}x^{M-I_2}, \end{aligned}$$

we find that

$$y_{i_0}x^{I_0} \otimes y^{J_0}x^{M-I_0} \equiv y_{i_2}x^{I_2} \otimes y^{J_2}x^{M-I_2}$$

if $J_0 + (i_0) = J_2 + (i_2)$ and if there exists an L with $L \leq M$, $L - I_0 \geq 0$ and $L - I_2 \geq 0$. Here we choose J_1 and i_1 in the following way: take $i_1 = \max(i_0, i_2)$ and take $J_1 = J_\alpha$ if $i_1 = i_\alpha$, $\alpha \in \{0, 2\}$. Notice that $|L| = \langle d, J_1\rangle + k$.

If $J_0 - J_2 = (i_2) - (i_0)$ and $I_0 - I_2 = (k_0) - (k_2)$ (i.e., I_0 and I_2 also differ by one change of index), we can choose L with $L - I_0 \geq 0$ and $L - I_2 \geq 0$ if $|L| > |I_0|$ and $|L| > |I_2|$, i.e., if

$$\langle d, J_1\rangle + k > \max(d_{i_0}, d_{i_2}).$$

By construction this holds if $\langle d, J\rangle + k > 0$, where we set $J = J_1 - (i_0)$ if $i_1 = i_2$ and $J = J_1 - (i_2)$ if $i_1 = i_0$.

By transitivity we can show the existence of L if I_0 and I_2 differ by more than one change of index. Hence

$$y_{i_0} x^{I_0} \otimes y^{J_0} x^{M-I_0} \equiv y_{i_2} x^{I_2} \otimes y^{J_2} x^{M-I_2}$$

if $J_0 + (i_0) = J_2 + (i_2)$, $M \geq I_0$, $M \geq I_1$ and $\langle d, J\rangle + k > 0$ ($J = J_0 - (i_2) = J_2 - (i_0)$). If

$$\begin{aligned} h(\sum_{i,I,K,L} c_{i,I,K,L} \;\; y_i x^I \otimes y^K x^L) &= \sum_{i,I,K,L} c_{i,I,K,L} \;\; y^{K+(i)} x^{I+L} \\ &= \sum_{(J,M)} \sum_{\substack{(i,I,K,L) \\ K+(i)=J, I+L=M}} c_{i,I,K,L} \;\; y^J x^M = 0, \end{aligned}$$

then

$$\sum_{\substack{(i,I,K,L) \\ K+(i)=J, I+L=M}} c_{i,I,K,L} = 0$$

for all pairs (J, M), hence

$$\sum_{i,I,K,L} c_{i,I,K,L} \;\; y_i x^I \otimes y^K x^L \equiv 0 \mod (\operatorname{im} g).$$

$\square$

Remark 3.5. The proof of Lemma 3.4 is based on the proof of the symmetrizer lemma for projective hypersurfaces by Donagi and Green [D-G]. It is possible to prove Lemmas 3.4 and 3.7 in a different way, using the abstract definition of the Jacobi ring from Chapter 1 and Castelnuovo–Mumford regularity.

Corollary 3.6. *Suppose that $n \geq 2$ and $p \geq 2$. The complex*

$$\bigwedge^2 S_{1,0} \otimes S_{p-2,d(X)} \xrightarrow{g} S_{1,0} \otimes S_{p-1,d(X)} \xrightarrow{h} S_{p,d(X)}$$

is exact at the middle term if condition $(*)$ of Lemma 3.3 is satisfied.

Proof: Apply Lemma 3.4 with $k = d(X)$. $\square$

Lemma 3.7. *Suppose that $n \geq 2$ and $p \geq 2$. If the conditions $(*)$ and $(**)$ of Lemma 3.3 are satisfied, the map*

$$S_{1,0} \otimes J_{p-1,d(X)} \longrightarrow J_{p,d(X)}$$

is surjective.

Proof: As $\deg(\partial F/\partial x_k) = (1,-1)$ $(0 \leq k \leq 2m+r)$ and $\deg(\partial F/\partial y_i) = (0, d_i)$ $(0 \leq i \leq r)$, it suffices to show that the map μ' that appears in the commutative diagram

$$\begin{array}{ccc} S_{1,0} \otimes S_{p-a-1,d(X)-b} \otimes J_{a,b} & \xrightarrow{\mu'} & S_{p-a,d(X)-b} \otimes J_{a,b} \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu'} \\ S_{1,0} \otimes J_{p-1,d(X)} & \xrightarrow{\mu} & J_{p,d(X)} \end{array}$$

is surjective if $(a,b) = (1,-1)$ and if $(a,b) = (0,d_i)$ $(i = 0,\ldots,r)$. If $(a,b) = (1,-1)$, then μ' is surjective if $\langle d, J\rangle + d(X) + 1 \geq 0$ for all J with $|J| = p-2$; this follows from condition $(*)$.

If $(a,b) = (0,d_i)$ $(i = 0,\ldots,r)$, then μ' is surjective if

$$\langle d, J\rangle + d(X) - d_i \geq 0$$

for all J with $|J| = p-1$. This follows from the condition $(**)$. □

Corollary 3.8. *Suppose that $m \geq 2$ and $d_0 \geq \ldots \geq d_r$. Then $\delta\nu = 0$ provided that*

(1) $d_0 + \ldots + d_r + (m-2)d_r \geq 2m + r + 2$

(2) $d_1 + \ldots + d_r + (m-1)d_r \geq 2m + r + 1$.

Proof: This follows from Lemmas 3.1 and 3.3. □

Note that the first condition in Corollary 3.8 is implied by the second one, unless $d_0 = \ldots = d_r$.

Lemma 3.9. *If the conditions* (1) *and* (2) *of Corollary 3.8 are satisfied, the normal function ν has flat local liftings that are unique up to sections of $H_{\mathbb{Z}}$.*

Proof: It follows from (1) and (2) that $\delta\nu = 0$, hence ν has flat local liftings. If $\tilde{\nu}$ and $\tilde{\nu}'$ are two flat liftings of ν over an open set $U_0 \subset T$, we can write

$$\tilde{\nu} - \tilde{\nu}' = \varphi + \lambda$$

where $\varphi \in H^0(U_0, \mathcal{F}^m)$ and $\lambda \in H^0(U_0, H_{\mathbb{Z}})$. Since $\nabla\varphi = \nabla(\tilde{\nu} - \tilde{\nu}') = 0$, it suffices to show that the map

$$\nabla : \mathcal{F}^m \to \Omega^1_T \otimes \mathcal{F}^{m-1}$$

is injective. By duality, it suffices to show that for all $t \in T$ the map

$$T \otimes H^{2m-p,p-1}(X_t) \to H^{2m-p-1,p}(X_t)$$

is surjective for $p \geq m$. This follows if the map

$$S_{1,0} \otimes S_{p-1,d(X)} \to S_{p,d(X)}$$

is surjective, and by Lemma 3.2 this holds if $\langle d, J\rangle + d(X) > 0$ for all J with $|J| = p - 1$. This follows from condition (1). $\square$

If the conditions (1) and (2) of Corollary 3.8 are satisfied, then the normal function ν is torsion. This is proved using a monodromy argument, which is taken from [V3, Lecture 4].

Lemma 3.10. *If ν has flat local liftings that are unique up to sections of $H_{\mathbb{Z}}$, then $\nu \in H^0(T, \mathcal{J}^m_h)$ is a torsion section of $\mathcal{J}$.*

Proof: Let $\tilde{\nu}$ be a flat local lifting of ν in an open neighbourhood of $t_0 \in T$. We have to show that $\tilde{\nu}(t_0) \in H^{2m-1}(X_0, \mathbb{Q})$. To this end we take a loop $\gamma : [0, 1] \to T$ based at t_0 and cover it by simply connected open sets U_α ($\alpha = 1, \ldots, k$) such that ν has a flat lifting ν_α on U_α. For all α, $\beta \in \{1, \ldots, k\}$ we have $\nu_\alpha - \nu_\beta = \lambda_{\alpha\beta}$ for some $\lambda_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, H_{\mathbb{Z}})$; hence we can modify ν_2 by λ_{12} to obtain $\nu_1 = \nu_2$ on $U_1 \cap U_2$. Proceeding in this way on $U_2 \cap U_3, \ldots, U_{k-1} \cap U_k$, we find a new flat lifting $\hat{\nu}$ of ν in $\gamma(1)$. Let $\rho : \pi_1(T, t_0) \to \operatorname{Aut} H^{2m-1}(X_0, \mathbb{C})$ be the monodromy representation. By definition we have

$$\rho(\gamma)(\tilde{\nu}(t_0)) - \tilde{\nu}(t_0) = \hat{\nu}(t_0) - \tilde{\nu}(t_0),$$

and by assumption this element belongs to $H^{2m-1}(X_0, \mathbb{Z})$.

Claim: If $\eta \in H^{2m-1}(X_0, \mathbb{C})$ and $\rho(\gamma)(\eta) - \eta \in H^{2m-1}(X_0, \mathbb{Z})$ for all $\gamma \in \pi_1(T, t_0)$, then $\eta \in H^{2m-1}(X_0, \mathbb{Q})$.

Note that the proof will be finished if we verify this Claim. To this end, we view X_0 as a hyperplane section of a smooth complete intersection $Y_0 \subset \mathbb{P}^{2m+r+1}$ of dimension $2m$ and multidegree $(d_0, \ldots, d_r)$. Set $L = \mathcal{O}_Y(1)$. The linear system $|L|$ corresponds to a projective linear subspace of $\mathbb{P}H^0(\mathbb{P}^n, E)$. Let $\Delta_L \subset |L|$ be the discriminant locus, and define $U_L = |L| \setminus \Delta_L$. It is known that $\Delta_L \subset |L|$ is an irreducible hypersurface (cf. [B-S, Lemma 1.6.5]). Choose a Lefschetz pencil $\mathbb{P}^1 \subset |L|$ of hyperplane sections of Y_0 that passes through the point $0 \in |L|$, and denote the discriminant locus in $\mathbb{P}H^0(\mathbb{P}^n, E)$ by Δ_E. As $\Delta_L = \Delta_E \cap |L|$, it follows that $\mathbb{P}^1 \cap \Delta_E = \mathbb{P}^1 \cap \Delta_L = \{t_1, \ldots, t_k\}$ is a finite set of points. The fundamental group of $U_L \cap \mathbb{P}^1 = \mathbb{P}^1 \setminus \{t_1, \ldots, t_k\}$ has standard generators γ_i winding once around t_i. Let $\delta_i \in H^{2m-1}(X_0, \mathbb{Z})$ be the vanishing cocycle associated to γ_i. Since $g_*\pi_1(T, t_0) \subset \pi_1(U, 0)$ has finite index, N say, we have $\gamma_i^N = g_*\tilde{\gamma}_i$ for $i = 1, \ldots, k$. According to the Picard–Lefschetz formula, the action of $\tilde{\gamma}_i$ via the monodromy representation is given by

$$\rho(\tilde{\gamma}_i)(\eta) = \eta \pm N\langle \eta, \delta_i \rangle \delta_i.$$

Hence we find that $\rho(\tilde{\gamma}_i)(\eta) - \eta = \pm N\langle \eta, \delta_i \rangle \delta_i \in H^{2m-1}(X_0, \mathbb{Z})$ for $i = 1, \ldots, k$. Thus $\langle \eta, \delta_i \rangle \in \mathbb{Q}$ for $i = 1, \ldots, k$. The pairing

$$\langle \, , \rangle : H^{2m-1}(X_0, \mathbb{Q}) \times H^{2m-1}(X_0, \mathbb{Q}) \to \mathbb{Q}$$

is non–degenerate over $\mathbb{Q}$, and induces an isomorphism

$$H^{2m-1}(X_0, \mathbb{Q}) \xrightarrow{\sim} \mathrm{Hom}(H^{2m-1}(X_0, \mathbb{Q}), \mathbb{Q}))$$

sending an element $\alpha \in H^{2m-1}(X_0, \mathbb{Q})$ to $\langle \alpha, - \rangle$. As the vanishing cocycles $\delta_1, \ldots, \delta_k$ generate $H^{2m-1}(X_0, \mathbb{Q})$ (see for instance [V3, Lecture 4, 2.3] or [DK, Exposé XVIII, 6.6.1]), it follows that $\langle \eta, \lambda \rangle \in \mathbb{Q}$ for all $\lambda \in H^{2m-1}(X_0, \mathbb{Q})$; hence $\eta \in H^{2m-1}(X_0, \mathbb{Q})$. □

4 Main result

We formulate and prove the main result of this Chapter, which extends the aforementioned theorem of Green–Voisin to the case of complete intersections in projective space.

Theorem 4.1. *Let $X = V(d_0, \ldots, d_r) \subset \mathbb{P}^{2m+r}$ be a smooth complete intersection of odd dimension $2m-1$ $(m \geq 2)$ and multidegree $(d_0, \ldots, d_r)$ $(d_0 \geq \ldots \geq d_r,\ d_i \geq 2$ for $i = 0, \ldots, r)$. If X is very general, then the image of the Abel–Jacobi map*

$$\psi_X : \mathrm{CH}^m_{\mathrm{hom}}(X) \to J^m(X)$$

is contained in the torsion points of $J^m(X)$, unless we are in one of the following cases:

(i) $(r=0)$ $X = V(d) \subset \mathbb{P}^4$ $(3 \leq d \leq 5)$, $X = V(3) \subset \mathbb{P}^6$, $X = V(3) \subset \mathbb{P}^8$.

(ii) $(r=1)$ $X = V(3,3) \subset \mathbb{P}^5$.

(iii) $(r=1)$ $X = V(d,2) \subset \mathbb{P}^{2m+1}$, $d \geq 2$, $m \geq 2$.

(iv) $(r=2)$ $X = V(d,2,2) \subset \mathbb{P}^{2m+2}$, $d \geq 2$, $m \geq 2$.

(v) $(r=3)$ $X = V(2,2,2,2) \subset \mathbb{P}^{2m+3}$, $m \geq 2$.

Proof: We have seen that if $X_0 = V(d_0, \ldots, d_r) \subset \mathbb{P}^{2m+r}$ is a very general complete intersection, every cycle $Z_0 \in Z^m_{\mathrm{hom}}(X_0)$ can be spread out to a relative cycle $Z_T \in Z^m_{\mathrm{hom}}(X_T/T)$ after taking a finite étale covering $T \to U$ of the parameter space. If the conditions (1) and (2) of Corollary 3.8 are satisfied, the normal function $\nu \in H^0(T, \mathcal{J}^m_h)$ associated to Z_T is torsion by Lemmas 3.9 and 3.10. Note that this is the case if

$$(m+r-1)d_r \geq 2m+r+2 = 2(m+r-1)+4-r,$$

that is, if

$$d_r \geq 2 + \frac{4-r}{m+r-1}.$$

For $r=0$ this condition is

$$d_0 \geq 2 + \frac{4}{m-1}.$$

This is the result for hypersurfaces of odd degree in projective space obtained by Green and Voisin. The only exceptions are the ones listed in (i); see [G2]. Note that the Abel–Jacobi map is trivial for quadric hypersurfaces, since their intermediate Jacobians vanish.

For $r \geq 1$, $m \geq 2$ we have $\frac{4-r}{m+r-1} \leq 2$. Therefore we are done if $d_r \geq 4$, and it remains to check the cases $d_r = 2$ and $d_r = 3$.

Case 1. $d_r = 2$

(1) $d_0 + \ldots + d_{r-1} + 2m - 2 \geq 2m + r + 2$

(2) $d_1 + \ldots + d_{r-1} + 2m \geq 2m + r + 1$.

Since $d_i \geq 2$ for $i = 0, \ldots, r$, condition (1) is always satisfied if $r \geq 4$; condition (2) is always satisfied if $r \geq 3$. We check the cases $r = 1$, $r = 2$ and $r = 3$ separately:

$*$ $r = 1$

If $(d_0, d_1) = (d, 2)$, then the condition (1) is satisfied if $d_0 \geq 5$, but (2) is never satisfied.

$*$ $r = 2$

(1) $d_0 + d_1 \geq 6$

(2) $d_1 \geq 3$.

For $(d_0, d_1, d_2) = (d, 2, 2)$, $d \geq 2$, the condition (2) is never satisfied. If $d_1 \geq 3$, then (1) and (2) are satisfied.

$*$ $r = 3$

(1) $d_0 + d_1 + d_2 \geq 7$

(2) $d_1 + d_2 \geq 4$.

We see that condition (2) is always satisfied; condition (1) is satisfied unless $(d_0, d_1, d_2, d_3) = (2, 2, 2, 2)$.

Case 2. $d_r = 3$

(1) $d_0 + \ldots + d_{r-1} + 3m - 3 \geq 2m + r + 2$

(2) $d_1 + \ldots + d_{r-1} + 3m \geq 2m + r + 1$.

As in this case $d_i \geq d_r = 3$ for $i = 0, \ldots, r$, condition (1) is satisfied if $m + 2r \geq 5$ and condition (2) is satisfied if $m + 2r \geq 4$. Hence (1) and (2) are satisfied if $m \geq 2$ and $r \geq 2$. The only remaining case is $m = 2$, $r = 1$:

(1) $d_0 + d_1 \geq 7$

(2) $2d_1 \geq 6$.

Both conditions are satisfied unless $(d_0, d_1) = (3, 3)$. □

Remark 4.2. Let us consider the exceptional cases (i)–(v):

(i) The cubic and quartic threefold are Fano threefolds that contain a positive–dimensional family F of lines; in both cases, the Abel–Jacobi map $\mathrm{Alb}(F) \to J^2(X)$ is surjective (cf. [Ty], [C-G] and [B-M]). The cubic fivefold $X = V(3) \subset \mathbb{P}^6$ contains a family F of 2–planes; Collino [Co] showed that $\mathrm{Alb}(F) \xrightarrow{\sim} J^3(X)$. For a very general quintic threefold $X = V(5) \subset \mathbb{P}^4$, the image of the Abel–Jacobi map is non–torsion; see [Gr] and [C-C]. Clemens [C] showed that the image of the Abel–Jacobi map is not even finitely generated; his proof is based on monodromy arguments. Voisin [V2] has given a different proof of this statement using infinitesimal methods. The image of the Abel–Jacobi map is also not finitely generated for a very general cubic sevenfold $X = V(3) \subset \mathbb{P}^8$; see [A-C].

(ii) For a very general intersection of two cubics $X = V(3,3) \subset \mathbb{P}^5$, the image of ψ_X is not finitely generated [Ba-MuSt].

(iii) This case is covered by the following result:

Theorem. *Let Y be a smooth projective variety of even dimension $2m$, and let L be a very ample line bundle on Y. Suppose that $X \in |L|$ is a general smooth divisor. If*

(i) $H^{2m-1}_{\mathrm{var}}(X) \neq 0$

(ii) $\mathrm{im}\, \mathrm{cl}_{Y,\mathbb{Q}} \cap H^{2m}_{\mathrm{pr}}(Y, \mathbb{Q}) \neq 0$

then $\operatorname{im}\psi_{X,\mathbb{Q}} \neq 0$.

This result is essentially due to Griffiths, N. Katz and Zucker; see [DK, Exposé XVIII, Cor. 5.8.7]. Let $U \subset |L|$ be the smooth part, and let $V_{\mathbb{Q}}$ be the local system of variable cohomology. Let $\mathbb{P}^1 \subset |L|$ be a Lefschetz pencil, with smooth part $U_0 = U \cap \mathbb{P}^1$. Katz shows that there is an injective map

$$H^{2m}_{\mathrm{pr}}(Y,\mathbb{Q}) \to H^1(U_0, V_{\mathbb{Q}}).$$

As this map sends the class $[Z]$ to the cohomological invariant $\partial\nu_Z$ of the associated normal function ν_Z (see [Z, Prop. 3.9]), the desired statement follows. Note that we can replace (ii) by $\mathrm{Hdg}^m_{\mathrm{pr}}(Y)_{\mathbb{Q}} \neq 0$, as it is possible to associate a normal function to a primitive Hodge class on Y (cf. [G3, Lecture 6]). In a similar way one can deduce the non–vanishing of the infinitesimal invariant $\delta\nu_Z$; see [MuSt].

If Y is a quadric of dimension $2m$ and $X = Y \cap V(d)$ is a smooth hypersurface section, the conditions of the previous theorem are satisfied if $Z = Z_1 - Z_2$ is the difference of two m–planes that belong to the different rulings of Y (note that X has nontrivial vanishing cohomology; see [DK, Exposé XI]).

(iv) This case can be handled in the same way as (ii). If $Y = V(2,2) \subset \mathbb{P}^{2m+2}$ is a complete intersection of two quadrics, it is known that Y contains exactly 4^{m+1} m–planes; the cohomology classes of the differences of these m–planes generate $H^{2m}_{\mathrm{pr}}(Y,\mathbb{Q})$. This result is due to Reid [R], see also [Mer].

(v) For $m = 2$, it is known that the image of ψ_X is non–torsion if X is very general. This follows from a result of Bardelli, and one can handle the cases where $m > 2$ by a generalization of his techniques. Details will appear elsewhere.

Remark 4.3.

(1) The cases (iii) and (iv) mentioned above are the only cases where the technique of Katz produces non–torsion normal functions, in view of the cohomological Noether–Lefschetz theorem (see [DK, Exposé XIX]). For

Calabi–Yau complete intersections and cubic sevenfolds (which can in some sense be interpreted as the 'mirrors' of rigid Calabi–Yau threefolds [A-C]) one uses a similar technique, based on Mark Green's Lemma (see [Kim] or [V3, Lecture 3]), which produces a countable union of 'good' components of the Noether–Lefschetz locus whose union is dense in the parameter space. For details, see [V2] or [Ba-MuSt].

(2) Let $V \subset U = \mathbb{P}H^0(\mathbb{P}^{2m}, \mathcal{O}_{\mathbb{P}}(d)) \setminus \Delta$ be a Zariski open subset of the complement of the discriminant locus for the family of hypersurfaces of degree d in $\mathbb{P}^{2m}$. If $d \geq 2$, then there are no nonzero normal functions that are defined over V. This is clear if $d = 2$; for $d \geq 3$ it is proved in [G-H, §3], using a result of N. Katz on cohomology with values in the local system of vanishing cohomology over a Lefschetz pencil (see [DK, Exposé XVIII, Th. 5.7]) and results of Zucker on normal functions defined over Lefschetz pencils (see [Z, Thm. (4.17) and Cor. (4.52)]). A similar argument applies in the case of complete intersections $X = V(d_0, \ldots, d_r)$ such that $d_i \geq 2$ for all $i = 0, \ldots, r$.

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